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OCCURRENCE SYSTEMS

A Note on the Metaphysics of Substitution

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In [1988b] Peter Aczel initiates a mathematical investigation of the intuitive notion of a *structured object* - a truly metaphysical project. Here I use his elegant approach to make a note on one distinction relevant to that project, namely, the one between something *occurring in* a structured object and its *having occurrences* in the object. Starting from his notion of an *X-form system,* defined below, where an 'occurs in' relation is present, I define a subclass of these, the *occurrence systems*, where the stronger notion of an occurrence is meaningful. I then investigate under what conditions an X-form system is isomorphic to an occurrence system, and some related issues. An instance of the present problem occupied me once before, in the form of a comparison between two notions of an *abstract logic*; cf. the bibliographical note at the end.

1. Preliminaries

DEFINITION 1.1: Let X be a class. An *X-form-system* is a triple $(A,C,\cdot)$ such that

A is a class

$C:A \to powX$

If $a \in A$ och $\sigma:Ca \to X$ then $\sigma\cdot a \in A$

$$C(\sigma\cdot a) = range(\sigma)$$
$$idCa\cdot a = a$$
$$\tau\cdot(\sigma\cdot a) = (\tau o \sigma)\cdot a\,, \text{ if } \tau:range(\sigma) \to X\,.$$

A functionp from A to A' is a*homomorphism* between the X-form systems $(A,C,\cdot)$ and $(A',C',\cdot')$ if for $a \in A$,

$Ca = C'pa$

$p(\sigma\cdot a) = \sigma\cdot' pa$;

p is an *isomorphism* if it is also a bijection. $(A,C,\cdot)$ is a *subsystem* of $(A',C',\cdot')$, in symbols $(A,C,\cdot) \subseteq (A',C',\cdot')$, if A is a *closed* subset of A', i.e., if $a \in A$ and $\sigma:Ca \to X$

imples $\sigma{\cdot}a \in A$, and also $\sigma{\cdot}a = \sigma{\cdot}'a$ and $C = C'|A$. Equivalently, $(A,C,\cdot) \subseteq (A',C',\cdot')$ iff $A \subseteq A'$ and the inclusion morphism from A to A' is a homomorphism.

Here are a few examples of X-form systems (all except 1.3 from Aczel [1988b]):

EXAMPLE 1.2: *Sets of parameters.* Let $A = powX$, $Ca = a$ for $a \in A$, and, for $\sigma{:}a \to X$, let $\sigma{\cdot}a = \{\sigma x : x \in a\}$.

EXAMPLE 1.3: *Numbers structured by primes.* Let $A = N = \{0,1,2,...,\}$, $Pr = \{p \in N :$ p is a prime$\}$. Each n has a unique prime factorization in the standard form $p_1^{k_1}\cdot...\cdot p_m^{k_m}$, with $k_i > 0$ for $i = 1,...,m$, and we let $Cn = \{p_1,...,p_m\}$. If $\sigma{:}Cn \to Pr$ let $\sigma{\cdot}n$ be the result of rewriting $(\sigma p_1)^{k_1}\cdot...\cdot(\sigma p_m)^{k_m}$ in standard form. It is easily checked that this is an Pr-form system.

EXAMPLE 1.4: *Sentential logic.* Let X be given, and let P be a set of atomic sentence symbols. Let $A = P \cup \{\neg x : x \in X\} \cup \{(x \wedge y) : x,y \in X\}$. Then let $Cp = \varnothing$ for p in P, $C\neg x = \{x\}$, and $C(x \wedge y) = \{x,y\}$. Finally, let $\sigma{\cdot}p = p$, $\sigma{\cdot}\neg x = \neg\sigma x$, $\sigma{\cdot}(x \wedge y) = (\sigma x \wedge \sigma y)$. $(A,C,\cdot)$ is clearly an X-form system.

EXAMPLE 1.5: *Another version of sentential logic.* In example 1.4 the 'forms' occurring are just negation, conjunction, and atomic formulas. We can also let each formula of the logic determine a 'form', shared only by formulas obtained from it by replacing sentential symbols. Thus, let L be the set of formulas in $\neg$ and $\wedge$ with sentential symbols from P, and for $\phi \in L$ let $C\phi$ be the set of atomic subformulas of ϕ; $\sigma{\cdot}\phi$ is defined in the obvious way. Then $(L,C,\cdot)$ is a P-form system.

We shall need to name some relations between elements of an X-form system. Let, for the rest of this section, $(A,C,\cdot)$ be a fixed X-form system.

DEFINITION 1.6: Define, for $a,b \in A$,

$a \leq b$ iff $b = \sigma\cdot a$ for some $\sigma{:}Ca \to X$

$a \equiv b$ iff $b = \sigma\cdot a$ for some 1-1 $\sigma{:}Ca \to X$

$a \text{ -- } b$ iff $a \leq b$ and $b \leq a$

$a \approx b$ iff $\exists a_1,\ldots,a_n \in A$ such that $a_1 = a$, $a_n = b$, and for $1 \leq i < n$, $a_i \leq a_{i+1}$ or $a_{i+1} \leq a_i$

$\leq$ is reflexive and transitive. $\equiv$, --, $\approx$ are equivalence relations such that $a \equiv b$ implies a -- b, which implies $a \approx b$. Further properties of these relations are stated below.

(1) If a and b have a lower bound in the $\leq$ pre-ordering, they also have an upper bound.

Proof: Suppose $\sigma\cdot a_0 = a$ and $\tau\cdot a_0 = b$. Take $z \in X$ and let $\sigma_0{:}range(\sigma) \to \{z\}$ and $\tau_0{:}range(\tau) \to \{z\}$. Then $\sigma_0 \circ \sigma = \tau_0 \circ \tau$, and thus $\sigma_0\cdot a = \tau_0\cdot b$.

(2) $a \approx b$ iff for some $c \in A$, $a \leq c$ and $b \leq c$.

Proof: One direction is immediate, and the other follows by an easy induction on n in the definition of $a \approx b$, using (1).

Let $D(a)$, for each $a \in A$, be the equivalence class of a under $\approx$. A subclass K of A is *connected* if $a,b \in K$ implies that there is $c \in K$ such that $a \leq c$ and $b \leq c$.

(3) Each $D \in A/{\approx}$ is a maximal connected subclass of A.

Proof: That D is connected follows from (2). To see that D is maximal connected, suppose $D \cup \{a\}$ is connected. Take $b \in D$. There is c such that $a \leq c$ and $b \leq c$. Thus, $c \in D$ and hence $a \in D$.

We shall be interested in the following uniqueness property of $(A,C,\cdot)$.

(U) If $a \in A$, $\sigma_i : Ca \rightarrow X$ for $i = 1,2$, and $\sigma_1 \neq \sigma_2$, then $\sigma_1 \cdot a \neq \sigma_2 \cdot a$.

Also, let ($U1$) be the same property for 1-1 σ_i. Of the previous examples, 1.4 and 1.5 satisfy (U), but not 1.2 or 1.3.

(4) If (U) holds then $a \text{ -- } b$ iff $a \equiv b$.

Proof: If $a = \sigma_1 \cdot b$ and $b = \sigma_2 \cdot a$, then $a = (\sigma_1 \circ \sigma_2) \cdot a$ and $b = (\sigma_2 \circ \sigma_1) \cdot b$. By ($U$) it follows that $id_{Ca} = \sigma_1 \circ \sigma_2$ and $id_{Cb} = \sigma_2 \circ \sigma_1$, so in particular, σ_1 is 1-1. [Note: If a and b are finite (i.e. if Ca and Cb are finite) it suffices to assume ($U1$), since $\sigma \cdot a = a$ implies that σ is onto Ca, hence 1-1 if Ca is finite.]

DEFINITION 1.6: A *cover* for $D(a)$ is a pair (C_0, G) where C_0 is a set and G a 1-1 function from $D(a)$ such that for all $b \in D(a)$, $Gb = G_b$ is a function from C_0 onto Cb, and if $\sigma : Cb \rightarrow X$, then $G_{\sigma \cdot b} = \sigma \circ G_b$.

An equivalent way of formulating the defining condition would be: G is a function from $D(a)$ such that if $b \in D(a)$ then G_b is a function from C_0 onto Cb, and if $b_i \in D(a)$ and $\sigma_i : Cb_i \rightarrow X$, $i = 1,2$, then $\sigma_1 \cdot b_1 = \sigma_2 \cdot b_2$ iff $\sigma_1 \circ G_{b_1} = \sigma_2 \circ G_{b_2}$.

A cover acts in a sense like a minimal element of $D(a)$. However, the existence of a cover is a much weaker property than the existence of a minimal element: we do not require *every* function from C_0 to X to yield an object in $D(a)$. The situation is clarified in Theorem 2.3.

(5) If $D(a)$ has a cover then (U) holds for all $b \in D(a)$.

Proof: If $b \in D(a)$, $\sigma_i : Cb \rightarrow X$ and $\sigma_1 \neq \sigma_2$, then $\sigma_i \cdot b \in D(a)$, and, since G_b is onto, $\sigma_1 \circ G_b \neq \sigma_2 \circ G_b$. Thus $\sigma_1 \cdot b \neq \sigma_2 \cdot b$.

2. Occurrence systems

An X-form system $(A,C,\cdot)$ has a notion 'occurs-in': x occurs in a iff $x \in Ca$. We want to capture the stronger notion of x having an 'occurrence' in a, where the same x may have different occurrences in an object. This requires objects to have a 'rigid structure' with fixed 'slots' for parameters, a structure that does not change under replacement of parameters. With each such structure is thus associated a set of slots. It is convenient, and natural, to represent such a structured object with a pair consisting of the structure and a function assigning a parameter to each slot.

The notion of a *signature* $\Omega = (\Omega,\nu)$ from universal algebra, where Ω is a class of 'symbols' (coding the structures in this case) and $\nu : \Omega \to powV$ gives the 'arity' (set of slots) of each symbol, can be used to express this idea:

DEFINITION 2.1: An *X-form occurrence system* (based on Ω) is an X-form system $(A,C,\cdot)$ such that

$$A \subseteq \{(a,f) : a \in \Omega \text{ and } f{:}\nu a \to X\}$$
$$C(a,f) = range(f)$$
$$\sigma\cdot(a,f) = (a,\sigma o f) .$$

We do not require that every (a,f) is in A. But since $(A,C,\cdot)$ is an X-form system, *if* $(a,f) \in A$ and $\sigma{:}C(a,f) \to X$ then $\sigma\cdot(a,f) \in A$. Moreover, all objects $\sigma\cdot(a,f)$ for σ a function from $C(a,f)$ to X are *distinct.*

In an occurrence system $(A,C,\cdot)$, every element of $D(a,f)$ $(= D((a,f)))$ has the form (a,g) for some g. Also,

(6) In an occurrence system every $D(a,f)$ has a cover.

Proof: Given $D(a,f)$, let $C_0 = \nu a$ and define G by $G_{(a,g)} = g$ if $(a,g) \in D(a,f)$. Thus $G_{(a,g)}$ is onto $C(a,g)$, and G is clearly 1-1. Also, if $\sigma{:}C(a,g) \to X$,

$$G_{\sigma\cdot(a,g)} = G_{(a,\sigma o g)} = \sigma \,o\, g = \sigma \,o\, G_{(a,g)} .$$

A 1-*X-form system* is defined in the same way as an X-form system, except that only 1-1 mappings $\sigma{:}Ca \to X$ are considered.

(7) (*i*) $(U1)$ is preserved under isomorphisms for 1-X-form systems.

(*ii*) (U) is preserved under isomorphisms.

(*iii*) If $p : (A_1,C_1,\cdot_1) \cong (A_2,C_2,\cdot_2)$, $a \in A_1$, and $D_1(a)$ has a cover, then $D_2(pa) = p[D_1(a)]$ has a cover.

Proof: Exercise.

We now consider the question of under what conditions an X-form system is isomorphic to an occurrence system.

THEOREM 2.1: A 1-X-form system $(A,C,\cdot)$ is isomorphic to a 1-occurrence system iff it satisfies $(U1)$.

Proof: The only-if-direction follows from (6), (5) and (7) (*i*). For the other direction, suppose $(A,C,\cdot)$ satisfies $(U1)$. Let $[a]$ be the equivalence class of a under $\equiv$. For each $K \in A/{\equiv}$, fix a_K such that $[a_K] = K$. Then define p as follows. Given $a \in A$, let $K = [a]$. By $(U1)$, there is a unique 1-1 $\sigma{:}Ca_K \to X$ such that $a = \sigma\cdot a_K$. Put $pa = ([a],\sigma)$. Then define

$$A' = range(p)$$
$$C'([a],\sigma) = range(\sigma)$$
$$\tau\cdot'([a],\sigma) = ([a],\tau\circ\sigma)\text{, when } \tau \text{ is a 1-1 function from } range(\sigma) \text{ to } X.$$

Note that $Ca = C'pa$, and that $p(\tau\cdot a) = ([a],\tau\circ\sigma) = \tau\cdot' pa$. It is then easy to check that $(A',C',\cdot')$ is a 1-X-form system. It is also an occurrence system, with $\Omega = A/{\equiv}$ and, for $K \in A/{\equiv}$, $\nu K = Ca_K$. Finally, p is 1-1, since if $a \neq b$ then either $[a] \neq [b]$, or $[a] = [b] = K$ so $a = \sigma_1\cdot a_K$ and $b = \sigma_2\cdot a_K$ where $\sigma_1 \neq \sigma_2$; in both cases $pa \neq pb$. *QED*

Unfortunately, the corresponding result for arbitrary X-form systems with (U) in place of $(U1)$ fails, as the next example shows.

EXAMPLE 2.2: Let X be a proper class and put

$$A = \{<x_\xi>_{\xi<\alpha} : 1 \le \alpha \in On, x_\xi \in X \text{ for } \xi < \alpha, \text{ and if } \alpha > 1, x_{\xi_1} \neq x_{\xi_2} \text{ for some } \xi_1,\xi_2 < \alpha\}$$

$$C<x_\xi>_{\xi<\alpha} = \{x_\xi : \xi < \alpha\}$$

If $\sigma{:}C<x_\xi>_{\xi<\alpha} \to X$ then

$$\sigma\cdot<x_\xi>_{\xi<\alpha} = <\sigma x_\xi>_{\xi<\alpha}, \text{ if } \alpha > 1 \text{ and } \sigma x_{\xi_1} \neq \sigma x_{\xi_2} \text{ for some } \xi_1,\xi_2 < \alpha,$$
$$= <\sigma x_0>, \text{ otherwise.}$$

Thus, $(A,C,\cdot)$ is almost the system of arbitrary wellordered sequences of parameters, but with the twist that at least two of the parameters in each sequence (of length > 1) must be distinct. It is easy to see that

(8) $(A,C,\cdot)$ is an X-form system satisfying (U).

Also, for any $a \in A$, $D(a) = A$ and, since X is a proper class,

(9) $\{|Ca| : a \in A\}$ has no upper bound.

But in an occurrence system clearly there is an upper bound to each $\{|Ca| : a \in D(b)\}$. Hence

(10) $(A,C,\cdot)$ is not isomorphic to an occurrence system

Using the notion of a cover, however, we can get a result corresponding to Theorem 2.1 for arbitrary X-form systems.

THEOREM 2.2: An X-form system $(A,C,\cdot)$ is isomorphic to an occurrence system iff every $D \in A/\approx$ has a cover.

Proof: The only-if-direction follows from (6) and (7) *(iii)*. For the if-direction, choose a cover for each $D(a)$. Define p as follows. Given $a \in A$, let (C_0,G) be the cover chosen for $D(a)$. Let $pa = (D(a),G_a)$, and define

$A' = range(p)$
$C'(D(a),G_a) = range(G_a)$
$\sigma\cdot'(D(a),G_a) = (D(a),\sigma o G_a)$, when $\sigma{:}range(G_a) \to X$.

Again, it is easily checked that $C'(D(a),G_a) = Ca$, that $p(\sigma\cdot a) = \sigma\cdot' pa$, and that $(A',C',\cdot')$ is an X-form system which is an occurrence system with $\Omega = A/\approx$ and $\nu(D(a)) = C_0$ as above. Also, since G above is 1-1, so is p. *QED*

Question: If $D \in A/\approx$, $\{|Ca| : a \in D\}$ has a supremum, and (U) holds for D, does it follow that D has a cover?

A natural class of occurrence systems are the *full* systems:

DEFINITION 2.3: An X-form occurrence system $(A,C,\cdot)$ based on Ω is *full*, if, for each $a \in \Omega$,

(*i*) $|\nu a| \leq |X|$ (if X is a set)

(*ii*) for every $f{:}\nu a \to X$, $(a,f) \in A$.

THEOREM 2.3: An X-form system $(A,C,\cdot)$ is isomorphic to a full occurrence system iff it satisfies (U) and every $D \in A/\approx$ has a minimal element a_D (i.e., such that $a_D \leq b$ for every $b \in D$).

Proof: For the only-if-direction, suppose $p : (A,C,\cdot) \cong (A',C',\cdot')$, where $(A',C',\cdot')$ is a full occurrence system with signature (Ω,ν). We know, by Theorem 2.2 and (5), that (U) holds for $(A,C,\cdot)$. To show that the second condition also holds, choose, for each $d \in \Omega$, a 1-1 function $h_d : \nu d \to X$. This is possible by (*i*) in the definition of a full occurrence system. By (*ii*) in that definition, $(d,h_d) \in A'$. Now take $D \in A/\approx$ and choose $a \in D$. Let $pa = (d,f)$. Then define

$$a_D = p^{-1}(d,h_d) .$$

We claim that a_D is a minimal element of D. To see this, take any $b \in D$. Then $pb = (d,g)$ for some g. This follows from

(*) If $b \leq a$ or $a \leq b$ then $b = (d,g)$ for some g.

Proof of (*): If $a \leq b$ then $\sigma \cdot a = b$ for some σ, and $p(\sigma \cdot a) = \sigma \cdot' pa = (d, \sigma o f)$. If $b \leq a$ then $\tau \cdot b = a$ for some τ. Let $pb = (d', g')$. Then $(d,f) = p(\tau \cdot b) = \tau \cdot' pb = (d', \tau o g')$, so $d = d'$.

Now, since h_d is 1-1, there is a (unique) $\sigma{:}range(h_d) \rightarrow X$ such that

$$\sigma \circ h_d = g .$$

Thus $(d,h_d) \leq (d,g)$ and it follows that

$$a_D \leq b .$$

This shows that a_D is minimal in D.

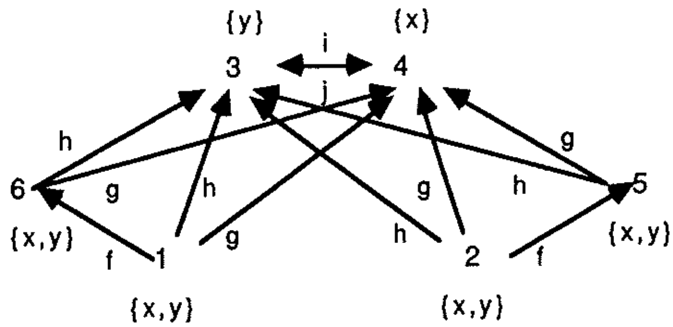
For the if-direction, we first show that each $D \in A/\approx$ has a cover. Let $C_0 = Ca_D$. For each $b \in D$ there is, by (U) and the fact that a_D is minimal, a unique $G_b{:}Ca_D \rightarrow Cb$ such that $G_b \cdot a_D = b$; let this define G. If $b_1 \neq b_2$, $b_1, b_2 \in D$, then $G_{b_1} \cdot a_D \neq G_{b_2} \cdot a_D$, so $G_{b_1} \neq G_{b_1}$, i.e., G is 1-1. Now suppose $\sigma{:}Cb \rightarrow X$. We have $G_{\sigma \cdot b} \cdot a_D = \sigma \cdot b$ and $G_b \cdot a_D = b$. So $(\sigma \circ G_b) \cdot a_D = \sigma \cdot (G_b \cdot a_D) = \sigma \cdot b$. Thus, by ($U$), $\sigma \circ G_b = G_{\sigma \cdot b}$, and (C_0, G) is a cover for D. We can then define an isomorphism p as follows. For every $a \in A$ there is a unique $\sigma{:}Ca_D \rightarrow X$ such that $a = \sigma \cdot a_D$, where $D = D(a)$. Let $pa = (D(a), \sigma)$ and set

$$A' = range(p)$$
$$C'(D(a),\sigma) = range(\sigma)$$
$$\tau \cdot'(D(a),\sigma) = (D(a), \tau o \sigma) \text{ , where } \tau{:}range(\sigma) \rightarrow X .$$

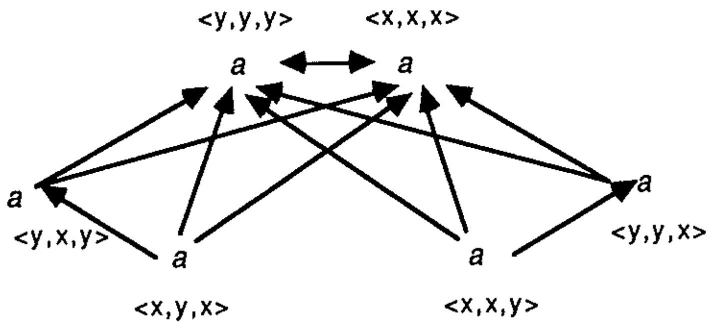
As before, $(A', C', \cdot')$ is an occurrence system with $\Omega = A/\approx$ and $\nu D = Ca_D$. It is also full, since for any $D \in \Omega$, $|\nu D| \leq |X|$ (if X is a set), and, whenever $\tau{:}\nu D \rightarrow X$, $\tau \cdot a_D \in D$ and $p(\tau \cdot a_D) = (D(\tau \cdot a_D), \tau) = (D, \tau) \in A'$. This completes the proof. *QED*

To illustrate the use of occurrence systems that are not full, we give the following simple example.

EXAMPLE 2.4: Which is the simplest $D \in A/\approx$, for some X-form system $(A, C, \cdot)$, which satisfies (U) but has no minimal element? It can be drawn as follows [arrows are the functions from the various Ca (identity functions are not drawn); distinct arrows must go to distinct elements because of (U); the diagram must 'commute']:



Here we have $X = \{x,y\}, fx = y, fy = x, gx = gy = x, hx = hy = y, ix = y, jy = x$ (and $D = A$). This D has a cover, and is isomorphic to an occurrence system, e.g.,



Here $\Omega = \{a\}$ and $|va| = 3 > |X|$. Also, e.g., $(a,<x,y,y>) \notin A$. Hence, this occurrence system satisfies none of the two requirements in Definition 2.3.

3. Extending to replacement systems

In X-form systems parameters are just replaced by other parameters. One wants to extend this to systems with more general substitution. Aczel considers two types of such systems, *X-substitution systems*, where parameters can be replaced by arbitrary objects, and *replacement systems*, where there are no parameters, but objects can be replaced by other objects. The latter notion has a very simple definition:

DEFINITION 3.1: A *replacement system* is an A-form system $(A,C,\cdot)$. Likewise, a *replacement occurrence system* based on Ω is an A-form occurrence system $(A,C,\cdot)$ based on Ω.

For example, if $X = powX$, Example 1.2 becomes a replacement system. Aczel's replacement system for sentential logic looks as follows.

EXAMPLE 3.2: Let L be the set of formulas in $\neg$ and $\wedge$ with sentential symbols from P, and define C by

$$Cp = \varnothing \text{ for } p \in P$$
$$C\neg\phi = \{\phi\}$$
$$C(\phi \wedge \psi) = \{\phi,\psi\}.$$

If $\sigma{:}C\phi \rightarrow L$, $\sigma{\cdot}\phi$ is defined in the obvious way. Then $(L,C,\cdot)$ is an L-form system.

Example 1.4 is (isomorphic to) an occurrence system, and Example 3.2 (to) a replacement occurrence system. Moreover, there is a fairly simple way to 'generate' 3.2 from 1.4: Put atomic formulas in the slots (instead of parameters), then put the thus obtained formulas in the slots, etc. The notion of an occurrence system has the advantage that this procedure can be applied in general, as will now be shown.

Fix a class X and a signature $\Omega = (\Omega,\nu)$ such that $|\nu a| \leq |X|$ (if X is a set) for $a \in \Omega$. Let $\Omega_{at} = \{a \in \Omega : \nu a = \varnothing\}$. Ω *has atoms* if $\Omega_{at} \neq \varnothing$.

If $a \in \Omega$ and f and g are two functions with domain νa, we say that f and g are *similar* if, for any $u,v \in \nu a$, $fu = fv \leftrightarrow gu = gv$. It is easy to show

(11) If f and g are similar, there is a bijection $\sigma{:}range(f) \rightarrow range(g)$ such that $g = \sigma\circ f$.

Let, for any class Y,

$$A_{\Omega,Y} = \{(a,f) : a \in \Omega \ \&\ f{:}\nu a \rightarrow Y\} .$$

For $A \subseteq A_{\Omega,Y}$, let $\mathbf{A}$ be the system $(A,C,\cdot)$, where

$C(a,f) \;=\; range(f)$
$\sigma\cdot(a,f) \;=\; (a,\sigma\circ f)$, if $\sigma{:}range(f) \rightarrow Y$

(this notation suppresses Y, but it will be clear from context which class is intended). Of course, $\boldsymbol{A}$ is a Y-form (occurrence) system only if it has the required closure property: if $(a,f) \in A$ and $\sigma{:}range(f) \rightarrow Y$,then $(a,\sigma\circ f) \in A$.

The (full) X-form occurrence system $A_{\Omega,X}$ is shown in Aczel [1988b] to be a *free* X-form system [which implies that it can be homomorphically mapped into any X-form system $\boldsymbol{B}$ which is adequate for Ω in the sense that each arity of a symbol of Ω is equinumerous with the set of parameters of some object of $\boldsymbol{B}$]. Clearly

(12) The X-form occurrence systems based on Ω are precisely the subsystems of $A_{\Omega,X}$.

Now, for any $A \subseteq A_{\Omega,X}$, define a class operator Γ_A as follows:

$\Gamma_A Y \;=\; \{\,(a,f):\; a \in \Omega, f{:}\nu a \rightarrow Y$, and for some $g{:}\nu a \rightarrow X$ which is similar to f, $(a,g) \in A\,\}$.

(13) Γ_A is *monotone* and *set-based* (cf. Aczel [1988a]). Furthermore, $A_1 \subseteq A_2 \;\Rightarrow\; \Gamma_{A_1} Y \subseteq \Gamma_{A_1} Y$.

By (13) we can then define

$I_A \;=\;$ the least fixed point of Γ_A above $\{(a,\varnothing) : a \in \Omega_{at}\}$
$\;=\;$ the smallest class Z s.t. $\{(a,\varnothing) : a \in \Omega_{at}\} \subseteq Z$ and $\Gamma_A Z \subseteq Z$.

Since $\Gamma_A \varnothing = \varnothing$, it follows that

(14) $I_A \neq \varnothing$ iff Ω has atoms.

Following Aczel [1988b] we define the *terms* T_Ω of Ω to be the smallest class Z containing $\{(a,\emptyset) : a \in \Omega_{at}\}$ such that if $a \in \Omega$ and $f{:}\mathrm{v}a \to Z$ then $(a,f) \in Z$.

(15) $T_\Omega \;=\; I_{A_{\Omega,X}}$

Proof: Note that if $a \in \Omega$ and $f{:}\mathrm{v}a \to Y$ we can always find a similar $g{:}\mathrm{v}a \to X$, since $|\mathrm{v}a| \le |X|$. Then $(a,g) \in A_{\Omega,X}$ and the result follows.

(16) If $A \subseteq A_{\Omega,X}$, then I_A is *closed*, i.e., $(a,f) \in I_A$ and $\sigma{:}range(f) \to I_A$ implies that $(a,\sigma\circ f) \in I_A$.

Proof: Suppose $(a,f) \in I_A$ with f similar to some $g{:}\mathrm{v}a \to X$ such that $(a,g) \in A$. If $\sigma{:}range(f) \to I_A$, it follows by (11) and the fact that $|range(f)| \le |\mathrm{v}a| \le |X|$ that we can find $\tau{:}range(g) \to X$ such that $\sigma\circ f$ is similar to $\tau\circ g$. Moreover, $(a,\tau\circ g) \in A$ since A is an X-form system. Thus $(a,\sigma\circ f) \in \Gamma_A(I_A) \subseteq I_A$.

By (16), I_A, defined as before (with I_A itself as the class of 'parameters') is a replacement occurrence system based on Ω. We have thus defined an operator $\boldsymbol{I}$ from the class of X-form occurrence systems based on Ω to the class of replacement occurrence systems based on Ω, by

$$\boldsymbol{I}(A) \;=\; I_A \,.$$

It follows from (13) that

(17) $\boldsymbol{I}$ is monotone, i.e., $A_1 \subseteq A_2 \subseteq A_{\Omega,X}$ implies that $\boldsymbol{I}(A_1) \subseteq \boldsymbol{I}(A_2) \subseteq T_\Omega$.

If B_1 is a Y-form occurrence system based on Ω and B_2 a Z-form occurrence system based on Ω, call B_1 and B_2 *similar*, if for each $(a,f) \in B_1$ there is a $(a,g) \in B_2$ with g similar to f, and vice versa. Also, say that Ω has *enough* atoms, if $|\mathrm{v}a| \le |\Omega_{at}|$ for all $a \in \Omega$. We collect some of the information obtained about extending X-form occurrence systems to replacement occurrence systems in a final theorem. The straight-forward proof, which uses the techniques introduced above, is omitted.

THEOREM 3.1: Suppose Ω has enough atoms, and $|va| \leq |X|$ (if X is a set) for $a \in \Omega$. Then the mapping $\boldsymbol{I}$ is 1-1, and if $\boldsymbol{A}$ is an X-form occurrence system based on Ω, $\boldsymbol{I}(\boldsymbol{A})$ is similar to $\boldsymbol{A}$.

As an example, the reader might want to check that the operator $\boldsymbol{I}$ applied to Example 1.4 yields Example 3.2 (when both are construed as occurrence systems). This is simpler than the general case, though, since both systems are full, and hence what we have is an instance of the fact that $\boldsymbol{I}(\boldsymbol{A}_{\Omega,X}) = \boldsymbol{T}_{\Omega}$.

4. Bibliographical note

My interest in questions of occurrence goes back to Westerståhl [1976], where, among other things, different notions of an *abstract logic* were compared. One such notion was from Barwise [1974]. If one takes just the *syntactic* part of that notion, lets X be a set of non-logical constants, forgetting the fact that these parameters are *sorted* by arities etc., then one obtains, roughly, something which is *almost* an instance of an X-form system. In fact, it could be called a *semi-X-form system*, which is defined as in Definition 1.1, except that it has the weaker condition that if $\sigma{:}Ca \rightarrow X$ then

$$C(\sigma{\cdot}a) \subseteq range(\sigma),$$

and accordingly the compositionality requirement that if $\tau{:}range(\sigma) \rightarrow X$ then

$$(\tau{\circ}\sigma){\cdot}a = (\tau|C(\sigma{\cdot}a)){\cdot}(\sigma{\cdot}a).$$

In a semi-X-form system parameters may actually 'disappear' during substitution. It is, however, not hard to prove that

(18) Under the condition (U), a semi-X-form system is in fact an X-form system.

Now, one comparison was between Barwise's notion of a logic and what I called a *sequence logic*, where sentences are sequences of symbols. But what I was after was the distinction which is the theme of the present note, i.e., that between a syntax with just an 'occurs in' relation, and one which has *occurrences*. This has nothing to do with sequences really, and after reading Aczel [1988b], the notion of an occurrence system suggested itself as a more elegant way of capturing the intuitive idea. Theorems 2.1 and 2.2 are, essentially, reformulations of (the syntactic side of) results in Westerståhl [1976].

References

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