A NOTE ON PARAMETERIZATION

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1 Introduction

The purpose of this note is to prove a result in Peter Aczel's theory of replacement systems (Aczel [1988, 1989]). The result is, roughly, that any such system A can be 'parameterized' by any class X of parameters (disjoint from A). The note is technical, but the notion of a replacement system is simple and intuitive, as is the idea behind the proof. In the rest of this introduction I will try to explain these ideas.

A *replacement system* A has a universe A of *objects*, which are *structured* in the sense that each object a has a set Ca of *components*, also in A, and there is a replacement operation $\cdot$, subject to certain natural conditions, such that for any mapping σ of Ca into A, $\sigma \cdot a$ is the object resulting by 'replacing' components in a according to σ.

A universe of sets can be viewed as a replacement system, where components are elements, and replacement of elements is the obvious operation. A different example is furnished by the formulas of propositional logic, say, where the components of a formula are its immediate subformulas, and replacement of subformulas is simply taken literally. In addition to these purely set theoretic and purely syntactic examples, the notion of a replacement system covers a variety of other kinds of structured objects. In particular, Aczel [1989] applies replacement systems to the objects dealt with in *situation theory* (cf. Barwise [1989]).

The components of an object may in turn have components, etc. (by the way, nothing prevents an object from being a component of itself), and the components of components of ... of a are called the *constituents* of a. A constituent of a marks a 'place' where replacement can be performed. Sometimes it is convenient to have special atoms, called *parameters*, at these places (an atom is an object with no components). For such 'parametric objects' we then have the operation of replacing constituent parameters by other (possibly parametric) objects. This substitution is distinct from simple replacement of components, but the two operations are related: substitution of parameters in a should give the same result as replacing the components of a by the results of substituting parameters in the components. When this holds, for a class of parameters X, we call the replacement system X-*parametric*.

The desired result, then, is that for any replacement system A and any disjoint class of parameters X, we can form an X-parametric replacement system $A[X]$ which extends A. Indeed, there is a uniform way to construct $A[X]$ which gives the *smallest* replacement system with these properties.

In particular cases, such as the two examples above, it is easy to see how to form $A[X]$, but we need to make it work for any A. The idea is simple. We would like to re-

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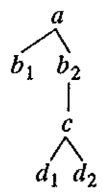
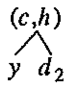
place components of objects in A by parameters in X, but A tells us nothing about this, since parameters are not in A. That is, if a is in A and f is a function with domain Ca and range (at least partly) in X, $f{\cdot}a$ is not defined. But we can try to 'represent' $f{\cdot}a$ by the *pair* (a,f). The set of components of this object is simply the range of f. This should take care of replacement of components of a by parameters. To get at the *constituents* of a, we must repeat the construction, 'replacing' in the same way components also by pairs (a,f), etc. Hopefully, when repetition comes to a halt (adds no more pairs), we shall have (up to isomorphism) the universe of the smallest replacement system above A of the required kind.

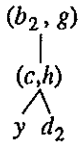
Here is an illustration. Let a be the structured object in A with components as depicted:



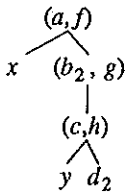
Suppose we wish to form the object where b_1 is replaced by x and d_1 by y, for $x,y \in X$. This takes three steps. At the first step we add



from c, where $hd_1 = y$, $hd_2 = d_2$. At the second step we add



from b_2, where $gc = (c,h)$. Finally, from a with $fb_1 = x$ and $fb_2 = (b_2,g)$ we add



which is the desired object.

Taken literally, though, this strategy yields too many objects. For example, consider an object $\rho{\cdot}c$ in A obtained by replacing in c above according to ρ, where $\rho d_1 = d_3$ and $\rho d_2 = d_4$. We may have $c \neq \rho{\cdot}c$, but if h' replaces in $\rho{\cdot}c$ in the same way as h replaces in c, $h'd_3 = y$, $h'd_4 = d_2$, clearly we want to *identify* (c,h) and $(\rho{\cdot}c\ ,\ h')$. So we need to find a suitable equivalence relation on such pairs, and add the equivalence class of (c,h) as a new object rather than (c,h) itself.

Since we want the *smallest* X-parametric replacement system extending A, we do not add (equivalence classes of) pairs (a, f) when the range of f is included in A, for then, $f{\cdot}a$ is already in A. We keep the original objects in A and must thus ensure, among other things, that if σ replaces all components in (the equivalence class of) (a, f) with objects in A, the result is also in A.

We noted that A need not be wellfounded (i.e., that there may exist an infinite sequence $a_0, a_1, a_2, \ldots$ of objects in A such that $a_{n+1} \in Ca_n$ for $n \geq 0$.) Consider the simple case of an object a whose sole component is a itself. Let $fa = x$. In contrast with a, the pair (a, f), obtained by 'replacing' a (in a) by the parameter x, is quite wellfounded: its only component is x, and x in turn has no components. In general, the new objects we construct will be wellfounded in the sense that if there is an infinite descending sequence $u_0, u_1, u_2, \ldots$ in $A[X]$, eventually the u_n are all in A.

With this we turn to the actual statement and proof of the result. The paper is self-contained, since all relevant definitions have been stated, but for further motivation and background the reader may consult Aczel [1988, 1989] and Westerståhl [1989].

2 Preliminaries

Replacement systems are special cases of what Aczel calls *X-form systems*, defined below. Let X be a class.

DEFINITION 2.1: (i) An *X-form system* is a triple $A = (A,C,\cdot)$ where A is a class, $C{:}A \rightarrow powX$, and if $a \in A$ and $\sigma{:}Ca \rightarrow X$ then $\sigma{\cdot}a \in A$ and

$$C(\sigma{\cdot}a) = range(\sigma)$$
$$id_{Ca}{\cdot}a = a$$
$$\tau{\cdot}(\sigma{\cdot}a) = (\tau\sigma){\cdot}a\ ,\ \text{ if } \tau{:}range(\sigma) \rightarrow X\ .$$

(ii) A *replacement system* is an A-form system $A = (A,C,\cdot)$.

In the first example mentioned above, A is a universe of sets, $Ca = a$ for $a \in A$, and $\sigma{\cdot}a = \{\sigma x : x \in a\}$ for $\sigma{:}Ca \rightarrow A$. In the second example, A is the set of formulas in propositional logic built up from atomic formulas $p, q, \ldots$ using $\neg$ and $\wedge$, C is defined by

$$Cp = \varnothing \quad \text{for } p \text{ atomic}$$
$$C(\neg\phi) = \{\phi\}$$
$$C(\phi \wedge \varphi) = \{\phi, \varphi\}\ ,$$

and the operation $\cdot$ by

$\sigma \cdot p \;=\; p$

$\sigma \cdot (\neg\, \phi) \;=\; \neg\, \sigma\phi$

$\sigma \cdot (\phi \wedge \varphi) \;=\; \sigma\phi \wedge \sigma\varphi\,.$

DEFINITION 2.2: Let $A_i = (A_i, C_i, \cdot_i)$ be X_i-form systems, $i = 1,2$. A_1 is a *subsystem* of A_2, in symbols $A_1 \subseteq A_2$, iff

$A_1 \subseteq A_2$ and $X_1 \subseteq X_2$

$C_1 \;=\; C_2 | A_1$

$\cdot_1$ is $\cdot_2$ restricted to A_1 and X_1, i.e., $a \in A_1$ and $\sigma{:}C_1 a \to X_1$ implies $\sigma \cdot_2 a = \sigma \cdot_1 a$.

In particular, this tells us what it means for one replacement system to be a subsystem of another.

The next definition generalizes Aczel's notion of a *homomorphism* between X-form systems to replacement systems.

DEFINITION 2.3: Let $A_i = (A_i, C_i, \cdot_i)$ be replacement systems, $i = 1,2$. A function p from A_1 to A_2 is a *homomorphism* if the following holds:

(i) $C_2 pa \;=\; \{px \,:\, x \in C_1 a\}$ for $a \in A_1$.

(ii) Suppose $a \in A_1$ and $\sigma{:}C_1 a \to A_1$, and let $\tau{:}C_2 pa \to A_2$ be any function such that $\tau px \;=\; p\sigma x$ for $x \in C_1 a$ (note that by (i), such a τ exists). Then $p(\sigma \cdot_1 a) \;=\; \tau \cdot_2 pa$.

It is easy to see that A_1 is a subsystem of A_2 iff the identity function id_{A_1} is a homomorphism, and that, if p is a homomorphism from A_1 to A_2 and q a homomorphism from A_2 to A_3, then qp is a homomorphism from A_1 to A_3.

A homomorphism p which is also 1-1 is called an *embedding*. In fact, this is the only case we shall need here. It is then easily seen that τ in condition (ii) above is uniquely determined by $\tau x \;=\; p\sigma p^{-1} x$ for $x \in C_2 pa$.

3 The induction step

For the rest of this paper, $A = (A, C, \cdot)$ is a fixed replacement system. As hinted at in section 1, the construction of an X-parametric replacement system above A will be inductive. In this section we give the details of the main step in the induction.

Fix two classes X and Z with $X \subseteq Z$ and $Z \cap A \;=\; \varnothing$ (Z will be the class of 'parametric objects' already added). Then set

$$F_Z \;=\; \{(a,f) :\; a \in A\,,\; f : Ca \to A \cup Z\,, \text{ and } range(f) \cap Z \neq \varnothing\}\,.$$

DEFINITION 3.1: For $(a,f)\,,\, (a',f') \in F_Z$ define

$(a,f) \leq_Z (a',f')$ iff there is a function $\rho{:}Ca \to Ca'$ such that $a' = \sigma \cdot a$ and $f = f'\rho$.

LEMMA 3.2: (i) $\leq_Z$ is reflexive and transitive.
(ii) If (a, f), (a', f') have a common lower $\leq_Z$-bound, they also have a common upper $\leq_Z$-bound.

Proof: (i) follows directly from the definition, so we prove (ii). Suppose then we have functions ρ , ρ' , h such that $a = \rho \cdot c$, $a' = \rho' \cdot c$, and $h = f\rho = f'\rho'$. h is a function from Cc, so there exists a 1-1 function g from $range(h)$ to Cc. Put $\tau = gf$ and $\tau' = gf'$. It follows that

$$\tau\rho = \tau'\rho' ,$$

which implies that $\tau \cdot a = \tau' \cdot a'$. Also

$$f = g^{-1}\tau \text{ and } f' = g^{-1}\tau' .$$

Thus, (a, f) , $(a', f') \leq_Z (\tau \cdot a, g^{-1})$.

Now we can define the equivalence relation we need on F_Z.

DEFINITION 3.3: $(a, f) \approx_Z (a', f')$ iff (a, f) and (a', f') have a common upper $\leq_Z$-bound.

LEMMA 3.4: (i) $\approx_Z$ is an equivalence relation.
(ii) If $(a, f) \approx_Z (a', f')$ then $range(f) = range(f')$.
(iii) If $(a, f) \approx_Z (a', f')$ and $\sigma : Crange(f) \to A \cup Z$, then
 (a) $range(\sigma) \cap Z = \varnothing$ implies $\sigma f \cdot a = \sigma f' \cdot a'$.
 (b) $range(\sigma) \cap Z \neq \varnothing$ implies $(a, \sigma f) \approx_Z (a', \sigma f')$.

Proof: For (i), the only problem is transitivity, but this follows easily from Lemma 3.2. (ii) is immediate. For (iii), suppose $(a, f) \leq_Z (c, h)$ via ρ and $(a', f') \leq_Z (c, h)$ via ρ', i.e.,

$$c = \rho \cdot a = \rho' \cdot a' ,\ f = h\rho ,\ \text{and } f' = h\rho' .$$

From this (a) follows directly. And in case (b), $(a, \sigma f) \leq_Z (c, \sigma h)$ via ρ, and $(a', \sigma f') \leq_Z (c, \sigma h)$ via ρ'.

DEFINITION 3.5:
$$E_Z = F_Z / \approx_Z$$
$$A_Z = A \cup X \cup E_Z$$

The next thing is to extend the operations C and $\cdot$ to A_Z. Let $[a, f]_Z$ be the equivalence class of (a, f).

DEFINITION 3.6: Define $C_Z u$, for $u \in A_Z$, by

$$C_Z x = \varnothing \quad \text{for } x \in X$$
$$C_Z a = Ca \quad \text{for } a \in A$$
$$C_Z[a,f]_Z = range(f) \quad \text{for } [a,f]_Z \in E_Z .$$

By Lemma 3.4 (ii), this is well defined. We have $C_Z u \subseteq A \cup Z$ for $u \in A_Z$.

DEFINITION 3.7: Suppose $u \in A_Z$ and $\sigma : C_Z u \to A \cup Z$. Define $\sigma \cdot_Z u$ as follows:

If $u \in X$ put $\sigma \cdot_Z u = u$.
If $u \in A$ and $range(\sigma) \cap Z = \varnothing$, put $\sigma \cdot_Z u = \sigma \cdot u$.
If $u \in A$ and $range(\sigma) \cap Z \neq \varnothing$, put $\sigma \cdot_Z u = [u,\sigma]_Z$.
If $u = [a,f]_Z$ and $range(\sigma) \cap Z = \varnothing$, put $\sigma \cdot_Z u = \sigma f \cdot a$.
If $u = [a,f]_Z$ and $range(\sigma) \cap Z \neq \varnothing$, put $\sigma \cdot_Z u = [a,\sigma f]_Z$.

That the last two cases of the definition are correct follows from Lemma 3.4 (iii). Let

$$\mathcal{A}_Z = (A_Z, C_Z, \cdot_Z) .$$

PROPOSITION 3.8: $\mathcal{A}_Z$ is an $A \cup Z$-form system such that $\mathcal{A} \subseteq \mathcal{A}_Z$.

Proof: Suppose $u \in A_Z$ and $\sigma : C_Z u \to A \cup Z$. We must show

(i) $\sigma \cdot_Z u \in A_Z$
(ii) $C_Z(\sigma \cdot_Z u) = range(\sigma)$
(iii) $id_{C_Z u} \cdot_Z u = u$.
(iv) if $\tau : range(\sigma) \to A \cup Z$, then $\tau \cdot_Z(\sigma \cdot_Z u) = (\tau\sigma) \cdot_Z u$.

(i) and (ii) are easy. As to (iii), the result is clear if $u \in X$. If $u \in A$, then $range(id_{Cu}) \cap Z = \varnothing$, so $id_{C_Z u} \cdot_Z u = id_{Cu} \cdot u = u$. And if $u = [a,f]_Z \in E_Z$, then $range(id_{C_Z u}) \cap Z \neq \varnothing$, so $id_{C_Z u} \cdot_Z u = [a, id_{C_Z u} f]_Z = u$. For (iv), finally, there are a number of cases to check, according to Definition 3.7.

Case 1: $u \in X$. The result is obvious.

Case 2: $u \in A$. There are 4 subcases, according to whether $range(\sigma) \cap Z$ and $range(\tau) \cap Z$ are empty or not. For example,

Subcase 2B: $range(\sigma) \cap Z = \varnothing$, $range(\tau) \cap Z \neq \varnothing$. Then

$$\tau \cdot_Z(\sigma \cdot_Z u) = \tau \cdot_Z(\sigma \cdot u) = [\sigma \cdot u, \tau]_Z$$
$$(\tau\sigma) \cdot_Z u = [u, \tau\sigma]_Z .$$

But $(u,\tau\sigma) \leq_Z (\sigma \cdot u, \tau)$ via σ.

Case 3: $u = [a,f]_Z \in E_Z$. Again there are four subcases, for example,

Subcase 3B: $range(\sigma) \cap Z = \varnothing$, $range(\tau) \cap Z \neq \varnothing$. Then

$$\tau \cdot_Z(\sigma \cdot_Z [a,f]_Z) = \tau \cdot_Z(\sigma f \cdot a) = [\sigma f \cdot a, \tau]_Z$$
$$\tau\sigma \cdot_Z [a,f]_Z = [a, \tau\sigma f]_Z .$$

But $(a,\tau\sigma f) \leq_Z (\sigma f \cdot a, \tau)$.

4 Digression

There is a certain arbitrariness in our definition of A_Z. We could instead have considered pairs (a,f) where f is 1-1, or where f is *constant on* A in the sense that if $fx \in A$ then $fx = x$. Intuitively, it seems that it would suffice to restrict attention to such replacement functions. This intuition is indeed correct, as we show in this section (which is not used in what follows). Let

$$F_Z{}^c = \{(a,f) \in F_Z : f \text{ is constant on } A\}$$
$$F_Z{}^1 = \{(a,f) \in F_Z : f \text{ is 1-1}\}$$
$$F_Z{}^{1,c} = \{(a,f) \in F_Z : f \text{ is 1-1 and constant on } A\}.$$

$E_Z{}^c$, $E_Z{}^1$, $E_Z{}^{1,c}$ are defined accordingly, and likewise $A_Z{}^c$, $A_Z{}^1$, $A_Z{}^{1,c}$. We first note that pairs in F_Z can be 'decomposed' using pairs of the more restricted kinds.

LEMMA 4.1: If $(a,f) \in F_Z$ there are functions $\tau{:}Ca \to A$ and $g{:}range(t) \to A{\cup}Z$ such that $(a,f) \leq_Z (\tau \cdot a,g)$ and $(\tau \cdot a,g) \in F_Z{}^c$ $(F_Z{}^1, F_Z{}^{1,c})$.

Proof: Let

$$D_Z = \{z \in Ca : fz \in Z\} \neq \varnothing$$
$$D_A = \{z \in Ca : fz \in A\}$$

Hence $Ca = D_Z \cup D_A$ and $D_Z \cap D_A = \varnothing$. Now let

$$\tau = id_{D_Z} \cup f{\restriction}D_A$$
$$g = f{\restriction}D_Z \cup id_{range(f{\restriction}D_A)}$$

Thus $\tau{:}Ca \to A$ and $g{:}C(\tau \cdot a) \to A{\cup}Z$ is constant on A. We have $g\tau = f$, so $(a,f) \leq_Z (\tau \cdot a,g)$. For the case of $F_Z{}^1$, we pick an element x_u in each set $f^{-1}\{u\}$ for $u \in range(f)$. Then put

$$\tau x = x_u \quad \text{for } x \in f^{-1}\{u\}$$
$$g x_u = u.$$

Again $g\tau = f$ and we are done. For $F_Z{}^{1,c}$ we apply the second construction to (a,f), and then the first to $(\tau \cdot a,g)$.

Next, $C_Z{}^c = C_Z$, etc., but the definitions of $\cdot_Z{}^c$ etc. do not work as they stand, since $\sigma{:}C_Z u \to A{\cup}Z$ need not be of required kind. This affects the third and fifth case of Definition 3.7. There, we apply the decompositions in the proof above, in the third case with σ for f, and in the fifth with σf for f. Thus, in the fifth case for $\cdot_Z{}^c$, for example, we put $\sigma \cdot_Z{}^c[a,f]_Z{}^c = [\tau \cdot a,g]_Z{}^c$, where $\tau{:}Ca \to A$, $g{:}C(\tau \cdot a) \to A{\cup}Z$ is constant on A, and $g\tau = \sigma f$. It now becomes a little more complicated to prove that this is well-defined, and that Lemma 3.8 (iv) holds. We illustrate with a typical case from the proof of the latter, w.r.t. $\cdot_Z{}^c$.

Subcase 3D: $range(\sigma)\cap Z \neq \emptyset$, $range(\tau)\cap Z \neq \emptyset$. Then

$$\tau\sigma\cdot_Z[a,f]_Z{}^c = [\tau_0\cdot a, g_0]_Z{}^c,$$

where g_0 is constant on A and

$$g_0\tau_0 = \tau\sigma f,$$

and

$$\tau\cdot_Z(\sigma\cdot_Z[a,f]_Z{}^c) = \tau\cdot_Z[\tau_1\cdot a, g_1]_Z{}^c = [\tau_2\tau_1\cdot a\,,\, g_2]_Z{}^c,$$

where g_1, g_2 are constant on A and

$$g_1\tau_1 = \sigma f$$

$$g_2\tau_2 = \tau g_1 .$$

Consider $(a, g_0\tau_0)$. We have $g_0\tau_0 = \tau\sigma f = \tau g_1\tau_1 = g_2\tau_2\tau_1$. It follows that $(a, g_0\tau_0) \leq_Z (\tau_0\cdot a, g_0)$ and $(a, g_0\tau_0) \leq_Z (\tau_2\tau_1\cdot a\,,\, g_2)$. Thus, $(\tau_0\cdot a, g_0)$ and $(\tau_2\tau_1\cdot a\,,\, g_2)$ have a common lower bound, and hence, by Lemma 3.2 (ii), a common upper bound. That is, $(\tau_0\cdot a, g_0) \approx_Z (\tau_2\tau_1\cdot a\,,\, g_2)$, as was to be proved.

The other cases also work out as expected (although with rather tedious details for $A_Z{}^{1,c}$), which establishes the first half of

PROPOSITION 4.2: $A_Z{}^c$, $A_Z{}^1$, $A_Z{}^{1,c}$ are all $A\cup Z$-form systems isomorphic to A_Z.

Proof: For the second half, consider $A_Z{}^c$. Define a function $p: A_Z{}^c \to A_Z$ by

$$pu = u \quad \text{for } u \in A\cup X$$

$$p[a,f]_Z{}^c = [a,f]_Z \quad \text{for } [a,f]_Z{}^c \in E_Z{}^c .$$

That p is 1-1 and onto follows easily using Lemma 4.1. It remains to show that if $u \in A_Z{}^c$ and $\sigma: C_Z u \to A\cup Z$, then $p(\sigma\cdot_Z{}^c u) = \sigma\cdot_Z pu$. This is clear except perhaps when $range(\sigma)\cap Z \neq \emptyset$. For example, if $u = [a,f]_Z{}^c$, we then have

$$p(\sigma\cdot_Z{}^c u) = p[\tau\cdot a, g]_Z{}^c = [\tau\cdot a, g]_Z, \text{ where } \sigma f = g\tau,$$

$$\sigma\cdot_Z pu = \sigma\cdot_Z[a,f]_Z = [a,\sigma f]_Z .$$

But $(a,\sigma f) \leq_Z (\tau\cdot a, g)$ and the result follows. The other cases are similar.

5 The construction of $A[X]$

Now we fix, in addition to A, a class X of parameters which is *completely disjoint* from A, in the sense that $X \cap (TC(a)\cup\{a\}) = \emptyset$ for all $a \in A$. The next definition, by induction over the ordinals, gives the 'parametric objects' we shall add to A.

DEFINITION 5.1:

$$X_0 = X$$

$$X_{\alpha+1} = X \cup E_{X_\alpha}$$

$$X_\lambda = \bigcup_{\xi<\lambda} X_\xi \qquad (\lambda \text{ limit ordinal})$$

Notice that all the X_α are disjoint from A.

LEMMA 5.2: If $\beta \leq \alpha$ then $X_\beta \subseteq X_\alpha$.

Proof: We use induction over α. The case when $\alpha \leq 1$ or α is a limit is obvious. So suppose $\alpha \geq 1$, $\gamma < \alpha+1$, and $u \in X_\gamma$; we must show $u \in X_{\alpha+1}$. We may clearly assume that γ is of the form $\beta+1$. If $u \in X$ then $u \in X_{\alpha+1}$, so suppose $u = [a,f]_{X_\beta}$ with $a \in A$ and $f{:}Ca \to X_\beta$. By induction hypothesis, $f{:}Ca \to X_\alpha$, so it is enough to show $[a,f]_{X_\beta} = [a,f]_{X_\alpha}$. We have $[a,f]_{X_\beta} \subseteq [a,f]_{X_\alpha}$ by the previous argument. Also, if $(b,g) \in [a,f]_{X_\alpha}$, i.e., $g{:}Ca \to X_\alpha$ and $(b,f) \approx_{X_\alpha} (a,f)$, then $(b,g) \in [a,f]_{X_\beta}$, since $range(g) = range(f) \subseteq X_\beta$.

Now put

$$A_\alpha = A \cup X_\alpha .$$

By Lemma 5.2 we see that

(1) $$A_{\alpha+1} = A \cup X_\alpha \cup E_{X_\alpha} = A \cup X \cup E_{X_\alpha} = A_{X_\alpha} .$$

DEFINITION 5.3: $\mathcal{A}_{\alpha+1} = (A_{\alpha+1}, C_{\alpha+1}, \cdot_{\alpha+1}) = \mathcal{A}_{X_\alpha} = (A_{X_\alpha}, C_{X_\alpha}, \cdot_{X_\alpha})$.

By Proposition 3.9, $\mathcal{A}_{\alpha+1}$ is an $\mathcal{A}_\alpha$-form system with $\mathcal{A} \subseteq \mathcal{A}_{\alpha+1}$. Moreover, it is easily checked that

(2) $\mathcal{A}_{\beta+1} \subseteq \mathcal{A}_{\alpha+1}$ when $\beta \leq \alpha$.

DEFINITION 5.4: Define $\mathcal{A}[X] = (A[X], C^*, \cdot^*)$ as follows:

(i) $$A[X] = \bigcup_{\alpha \in On} A_\alpha$$

(ii) For $u \in A[X]$, take α with $u \in A_{\alpha+1}$ and put $C^*u = C_{\alpha+1}u$.

(iii) If $u \in A[X]$ and $\sigma{:}C^*u \to A[X]$, we can find α such that $u \in A_{\alpha+1}$ and $range(\sigma) \subseteq A_\alpha$ (using the axiom of replacement). Put $\sigma \cdot^* u = \sigma \cdot_{\alpha+1} u$.

It follows from (2) that $\mathcal{A}[X]$ is well defined by (i) - (iii). We shall see that $\mathcal{A}[X]$ has the desired properties.

DEFINITION 5.5: A replacement system $\mathbf{B} = (B, C, \cdot')$ is *X-atomic*, if $X \subseteq B$ and $C'x = \emptyset$ for $x \in X$. $\mathbf{B}$ is called *X-parametric over A*,[1] if it is X-atomic, $A \subseteq B$, and every function $s{:}X \to B$ has a unique extension $\hat{s}{:}B \to B$ such that

(i) $\hat{s}|A = id_A$,

(ii) $\hat{s}a = (\hat{s}|C'a)\cdot' a$, for $a \in B - X$.

PROPOSITION 5.6: $A[X]$ is an X-parametric replacement system over A.

Proof: That $A[X]$ is an X-atomic replacement system is an immediate consequence of our previous results, as is the fact that $A \subseteq A[X]$. Take any function $s{:}X \to A[X]$. If we put

$$\hat{s}u = su, \text{ if } u \in X$$
$$\hat{s}u = u, \text{ if } u \in A$$
$$\hat{s}u = (\hat{s}|C^*u)\cdot^* u, \text{ if } u \text{ is of the form } [(a,f)]_{X_\alpha},$$

it is not hard to show that this actually gives an inductive definition of $\hat{s}|A_\alpha$ for each α, since, when $u \in A_{\alpha+1} - (A \cup X)$, $C^*u \subseteq A_\alpha$, so $(\hat{s}|C^*u)$ is already defined. Thus $\hat{s}$ is well defined, and clearly satisfies (i) and (ii) above. Also, a simple inductive argument shows that $\hat{s}$ is unique. Hence $A[X]$ is X-parametric.

From now on we drop the * in Definition 5.4, i.e., we extend the operations C and $\cdot$ to $A[X]$. We want to show that $A[X]$ is in a relevant sense the smallest X-parametric replacement system over A. First we note that inside any X-atomic extension of A, we can mimick the construction of $A[X]$. For let $\mathbf{B} = (B, C', \cdot')$ be such a replacement system, and define

$$B_0 = A \cup X$$
$$B_{\alpha+1} = X \cup \{\sigma\cdot' a : a \in A \text{ and } \sigma{:}Ca \to B_\alpha\}$$
$$B_\lambda = \bigcup_{\xi<\lambda} B_\xi \qquad (\lambda \text{ limit ordinal})$$

Then clearly $A \subseteq B_0 \subseteq B_1 \subseteq \ldots \subseteq B_\alpha \subseteq B_{\alpha+1} \subseteq \ldots \subseteq B$, and we put

$$B_{A,X} = \bigcup_{\alpha \in On} B_\alpha$$

Now it is easy to see that $B_{A,X}$ is *closed* in the sense that if $u \in B_{A,X}$ then $C'u \subseteq B_{A,X}$, and if $\sigma{:}C'u \to B_{A,X}$, then $\sigma\cdot' u \in B_{A,X}$. Therefore, the following definition makes sense.

[1] This is a slight variant of a notion introduced in Aczel [1989].

DEFINITION 5.7: If B is an X-atomic extension of A, let $B_{A,X}$ be the replacement system obtained by restricting B to $B_{A,X}$.

LEMMA 5.8: $A \subseteq B_{A,X} \subseteq B$, and $B_{A,X}$ is X-parametric over A.

Proof: We only need to check that $B_{A,X}$ is X-parametric over A, but this follows just as in the proof of Proposition 5.6.

We need one more lemma.

LEMMA 5.9: Suppose $A \subseteq B$, $a_i \in A$, $\sigma_i{:}Ca_i \to B$, $i = 1{,}2$, and $\sigma_1{\cdot}{'}a_1 = \sigma_2{\cdot}{'}a_2$. Then there are $\pi_i{:}Ca_i \to A$ and $h{:}range(\pi_1) \to B$ such that $\pi_1{\cdot}a_1 = \pi_2{\cdot}a_2$ and $\sigma_i = h\pi_i$, $i = 1{,}2$.

Proof: Take a 1-1 function $\tau{:}range(\sigma_1) \to Ca_1$, and let $\pi_i = \tau\sigma_i$. Then $\pi_1{\cdot}a_1 = \tau\sigma_1{\cdot}{'}a_1 = \tau{\cdot}{'}(\sigma_1{\cdot}{'}a_1) = \tau{\cdot}{'}(\sigma_2{\cdot}{'}a_2) = \tau\sigma_2{\cdot}{'}a_2 = \pi_2{\cdot}a_2$. Also, $\sigma_i = \tau^{-1}(\tau\sigma_i)$, so we can take $h = \tau^{-1}$.

We can now state our main result.

THEOREM 5.10: Let A be a replacement system and X a class completely disjoint from A. There is a canonical way to construct an X-parametric replacement system $A[X]$ over A, which furthermore has the property that for each X-atomic extension B of A there is a unique embedding from $A[X]$ to B which is the identity on $A \cup X$.

Proof: Only the second part remains to be proved. Given any X-atomic extension B of A, we define by induction a function $p{:}A[X] \to B_{A,X}$ such that $p|A_\alpha : A_\alpha \to B_\alpha$.
(i) If $u \in A_0 = A \cup X$, let $pu = u$.
(ii) Suppose $\alpha > 0$ and that for $\beta < \alpha$, p is defined on A_β such that $p|A_\beta : A_\beta \to B_\beta$. Take $u \in A_\alpha$. We may assume $u \in A_{\beta+1} = A \cup X \cup E_{X_\beta}$ with $\beta+1 \leq \alpha$. If $u \in A \cup X$, let $pu = u$. If $u = [a, f]_{X_\beta}$, then $pf : Ca \to B_\beta$ by induction hypothesis. Put

$$pu = pf{\cdot}{'}a \in B_{\beta+1} \subseteq B_\alpha .$$

(This is well defined, since one easily checks that $(a, f) \leq_{X_\beta} (c,h)$ implies $pf{\cdot}{'}a = ph{\cdot}{'}c$.)

First we show that p is 1-1. It suffices to check that if p is 1-1 on A_α, and if $u_i = [a_i, f_i]_{X_\alpha} \in A_{\alpha+1}$, $i = 1{,}2$, are such that $pu_1 = pu_2$, then $u_1 = u_2$. By assumption, $pf_1{\cdot}{'}a_1 = pf_2{\cdot}{'}a_2$. Thus, by Lemma 5.9, there are $\pi_i{:}Ca_i \to A$, such that $\pi_1{\cdot}a_1 = \pi_2{\cdot}a_2$, and there is a $h{:}range(\pi_1) \to B_{A,X}$ such that $pf_i = h\pi_i$, $i = 1{,}2$. By induction hypothesis, p is 1-1 on $range(f_i) \subseteq A_\alpha$. But this means that $(a_i, f_i) \leq_{X_\alpha} (\pi_i{\cdot}a_i , p^{-1}h)$, i.e., $u_1 = u_2$.

Next, we clearly have, for $u \in A[X]$,

(3) $C'pu = \{px : x \in Cu\}$.

Also,

(4) If $\sigma : Cu \to A[X]$ then $p(\sigma \cdot u) = p\sigma p^{-1} \cdot{}' pu$.

To check this, take α such that $u \in A_{\alpha+1}$ and $\sigma : Cu \to A_\alpha = A \cup X_\alpha$. There are a number of cases.

Case 1: $u \in X$. Then $pu = u$ and $\sigma \cdot u = p\sigma p^{-1} \cdot{}' u = u$.

Case 2: $u \in A$. Then $pu = u$ and $p|Cu = id_{Cu}$, so we must show

$$p(\sigma \cdot u) = (p\sigma) \cdot{}' u .$$

Case 2A: $range(\sigma) \cap X_\alpha = \varnothing$. Then $\sigma \cdot u \in A$ so $p(\sigma \cdot u) = \sigma \cdot u$, and $p\sigma = \sigma$ since $range(\sigma) \subseteq A$.

Case 2B: $range(\sigma) \cap X_\alpha \neq \varnothing$. Then $p(\sigma \cdot u) = p[a,\sigma]_{X_\alpha} = (p\sigma) \cdot{}' u$.

Case 3: $u = [a, f]_{X_\alpha}$.

Case 3A: $range(\sigma) \cap X_\alpha = \varnothing$. Using *Case* 2 we get $p\sigma p^{-1} \cdot{}' pu = p\sigma p^{-1} \cdot{}' (pf \cdot{}' a) = (p\sigma f) \cdot{}' a = p(\sigma f \cdot a) = p(\sigma \cdot u)$.

Case 3B: $range(\sigma) \cap X_\alpha \neq \varnothing$. Then $p\sigma p^{-1} \cdot{}' pu = (p\sigma f) \cdot{}' a = p[a, \sigma f]_{X_\alpha} = p(\sigma \cdot u)$.

Thus, p is an embedding from $A[X]$ to $B_{A,X}$. Moreover, if p' is any other embedding from $A[X]$ to $B_{A,X}$ such that $p' | A \cup X = id_{A \cup X}$, an easy inductive argument shows that $p = p'$. This completes the proof.

In terms of category theory, what we have shown is that $A[X]$ is an initial object in the category of X-atomic extensions of A, where the arrows are embeddings which are the identity on $A \cup X$. As usual, such initial objects are unique up to isomorphism.[2]

References

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Aczel, P., 1989, "Replacement systems and the axiomatisation of situation theory", abstract for the Asilomar conference on situation theory and its applications, April 1989.

[2] After completing a first version of this paper I learned from Peter Aczel that the main theorem is a special case of a so far unpublished result of his. Since the special case is of some interest for situation theory (cf. Westerståhl [1989]) and the method of proof is elementary, there may still be some point in its publication. I gratefully acknowledge that corresponding with Peter Aczel about replacement systems has been very helpful, and in particular enabled me to improve certain points in the presentation here.

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