ITERATED QUANTIFIERS*

Dag Westerståhl

Department of Philosophy, Stockholm University,
106 91 Stockholm, Sweden, westerstahl@philosophy.su.se

1. Introduction

This paper deals with a special kind of generalized quantifiers, called *iterations*. As expected, iterations are obtained by *iterating* quantifiers, of certain types. Equivalently, they are definable by (generalized) *quantifier prefixes*. This generalizes the notions of a quantifier prefix, and of prenex form, familiar from elementary logic, to logic with generalized quantifiers. Another motive for studying iterations is linguistic. A wide range of sentences in natural languages have truth conditions representable by means of iterations. When this is possible, the *scope relations* between noun phrases in the sentences are directly reflected in the corresponding prefix, by the left-right order. Scope *ambiguities* are accounted for by permutations of that order. Furthermore, there are other sentences, seemingly similar to the ones using iterations, whose truth conditions can be represented by other kinds of generalized quantifiers, but, on a closer look, *not* by iterations. Thus, it becomes of interest to know just when these other kinds of quantifiers are iterations, and when they are not. Several such questions will be addressed in this paper.

The first significant results on generalized quantifier prefixes were obtained by Edward Keenan. In fact, Part I of the present paper is my way of understanding his two main results in this field, the 'Reducibility Equivalence Theorem' in Keenan 1992 and the 'Generalized Linear Prefix Theorem' in Keenan 1993. Keenan usually writes with particular linguistic ap-

---

*This paper was written during a visit to the Institute for Logic, Language and Information at the University of Amsterdam in the autumn of 1992. I wish to thank the Institute for its hospitality, and several of its members for stimulating discussions. In particular, Johan van Benthem, Jaap van der Does, and Maarten de Rijke made useful comments on the first version of the paper. The present version has benefitted from valuable remarks by Dorit Ben-Shalom and by the editors of this volume.

plications in mind, but these theorems also have a purely logical interest. I will reformulate them in a setting more familiar to logicians, generalize them slightly, and bring out certain techniques which are implicit in their proofs. Part II contains a number of applications of these techniques to questions of definability of generalized quantifiers.

In more detail, the paper is organized as follows. In section 1, the iteration operation is defined for a suitable class of quantifiers, and motivated by a number of linguistic examples. Section 2 presents some useful properties of iterations, and section 3 contains (generalizations of) the two results by Keenan mentioned above. In section 4, the *convertible* iterations are characterized, i.e., those which are 'closed under converses', and as a corollary we also obtain necessary and sufficient conditions for a *resumption* (an ordinary monadic quantifier applied to n-tuples instead of individuals) to be an iteration. The main result of Section 5 gives a similar characterization for *branching* quantifiers, and section 6 one for *cumulations* (quantifiers rendering the so-called cumulative readings of certain sentences). Section 7 takes up the issue (raised in van Benthem 1989) of when a quantifier is a Boolean combination of iterations, and we prove, among other things, that the resumption of the quantifier ***most*** to pairs instead of individuals is not such a Boolean combination. Section 8, finally, lists some problems for further study.[1]

I
ITERATIONS AND THEIR PROPERTIES

2. Motivation and definitions

As usual, a (generalized) *quantifier of type* $<k_1,\ldots,k_n>$ ($k_i \geq 1$) is a functional $\boldsymbol{Q}$ which to each non-empty set M assigns a *quantifier* $\boldsymbol{Q}_M$ *of type* $<k_1,\ldots,k_n>$ *on* M, i.e., an n-ary relation between subsets of $M^{k_1},\ldots,M^{k_n}$, respectively. $\boldsymbol{Q}$ is *monadic* if $k_i = 1$, $i = 1,\ldots,n$, *polyadic* otherwise. $\boldsymbol{Q}$ is *simple* if $n = 1$.

To $\boldsymbol{Q}$ corresponds a *quantifier symbol* Q (of the same type), which acts as a variable-binding operator according to the following formation rule: if $\phi_1,\ldots,\phi_n$ are formulas and $x_{11},\ldots,x_{nk_n}$ are distinct variables, then

$$Qx_{11}\ldots x_{1k_1},\ \ldots\ ,x_{n1}\ldots x_{nk_n}(\phi_1,\ldots,\phi_n)$$

---

[1]The main results of this paper were announced, in weaker forms and without proofs, in Westerståhl 1992.

is a formula. By adding $\boldsymbol{Q}$ to elementary logic, with this formation rule and a corresponding additional clause in the definition of satisfaction, one obtains the *logic* $L(\boldsymbol{Q})$, and similarly $L(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)$. The formation rules and satisfaction clauses for the usual type <1> quantifiers $\forall$ and $\exists$ can be seen as instances of this.

Call the quantifier symbol followed by an appropriate string of variables, $Qx_{11}\ldots x_{1k_1}, \ldots ,x_{n1}\ldots x_{nk_n}$, a *quantifier expression*. Quantifier expressions with simple quantifier symbols, i.e., those applying to just one formula, can be iterated: put one more in front of a formula and you get a new formula. A (generalized quantifier) *prefix* is a finite string of simple quantifier expressions, with all variables distinct. If $\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k$ are simple, a formula of $L(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)$ is in *prenex form* if it has the form of a prefix (which may contain $\forall$ and $\exists$) followed by a quantifier-free formula.

Iterating quantifier expressions is one thing, iterating quantifiers is another, though of course related, thing. To see which kind of quantifiers we want to iterate, let us look at a few examples from natural language.

The canonical quantified English sentence has quantified subject and object noun phrases and a transitive verb, as in

(1) Most critics reviewed two books.

This can be formalized as a quantifier $\boldsymbol{Q}$ applied to three arguments, the set of critics (A), the set of books (B), and the relation denoted by *reviewed* (R); $\boldsymbol{Q}$ is thus of type <1,1,2>. But clearly it is more informative to represent the truth condition of (1) by means of the two familiar type <1,1> quantifiers ***most*** and ***two***. Indeed (suppressing the universe M),

$$\boldsymbol{Q}AB,R \quad \Leftrightarrow \quad \boldsymbol{most}\, A\{a\text{: } \boldsymbol{two}\, B\{b\text{: } Rab\}\}.$$

We will call $\boldsymbol{Q}$ the *iteration* of ***most*** and ***two***, and formalize (1) as

$$\boldsymbol{most}\cdot\boldsymbol{two}\ AB,R.$$

One advantage of this is that the other *reading* of (1), that there were two books such that most critics reviewed both of them, can now be represented as another iteration $\boldsymbol{two}\, B\{b\text{: } \boldsymbol{most}\, A\, \{a\text{: } Rab\}\}$, i.e.,

$$\boldsymbol{two}\cdot\boldsymbol{most}\ BA,R^{-1}$$

(note that we always take the first set argument to be linked to the first argument of the relation, and the second set argument to the second relation argument; hence the appearance of R^{-1} above).

There are more complex iterations. Consider

(2) Two boys gave more dahlias than roses to three girls.

Here three quantifiers are iterated, the first and the third of type <1,1>, but the second is the type <1,1,1> quantifier ***more-than*** (defined by ***more-than*** $ABC \Leftrightarrow |A{\leftrightarrow}C| > |B{\leftrightarrow}C|$), and the resulting quantifier has type <1,1,1,1,3>. We should have

$$\textbf{\textit{two}}\cdot\textbf{\textit{more-than}}\cdot\textbf{\textit{three}}\ ABCD,R$$
$$\Leftrightarrow\quad \textbf{\textit{two}}\ A\{a\text{: }\textbf{\textit{more-than}}\ BC\{b\text{: }\textbf{\textit{three}}\ D\{c\text{: }Rabc\}\}\};$$

this gives *one* reading of (2).[2]

It thus seems clear that we should be able, in principle, to iterate arbitrary monadic quantifiers. In fact, we will define iteration for an even larger class, which includes (certain) polyadic quantifiers as well, and which is *closed* under iteration. It is not surprising that this turns out to simplify the definition; after all, the iteration of two monadic quantifiers is polyadic. More interesting is the fact that this move also has a linguistic motivation. Keenan 1992 gives several examples involving ‘unreducible’ polyadic quantifiers, among them

(3) Every student criticized himself
(4) Every boy likes a different girl.

One reading of (4) uses the type <1,1,2> quantifier ***ED***, defined by

$$\textbf{\textit{ED}}\ AB,R \quad\Leftrightarrow\quad \forall a,b\in A(a \neq b \Rightarrow \exists c\in B(Rac\ \&\ \neg Rbc)),$$

and (3) uses the type <1,2> quantifier $\textbf{\textit{EH}}\ A,R \Leftrightarrow \forall a\in A\ Raa$. Now, although none of these are iterations, as Keenan shows, they can themselves be iterated with other quantifiers:

(5) Every student introduced himself to two professors
(6) Every boy gave different flowers to two girls.

For example, one reading of (6) should be rendered

$$\textbf{\textit{ED}}\cdot\textbf{\textit{two}}\ ABC,R \quad\Leftrightarrow\quad \textbf{\textit{ED}}\ AB,\{(a,b)\text{: }\textbf{\textit{two}}\ C\{c\text{: }Rabc\}\},$$

and the other reading is obtained by permuting the two quantifiers as before.

We are now ready to define iteration. Here is the relevant class of quantifiers.

---

[2] It is instructive at this point to work out just what this reading says, and which the other five readings are. Some of these may strike the reader as more natural for (2), whereas some will seem *very* unnatural. But if the latter are ruled out, this is mainly due to contingent facts about the relation of giving, and not, it seems to me, to any principled impossibility of these readings.

1.1. Definition. **CIT** is the class of quantifiers of types $<1,\ldots,1,k>$ with $m+1$ arguments, such that $m \geq 0$, $k \geq 1$, and if $m > 0$, then $m \geq k$. For obvious reasons, the first m arguments are called the *noun arguments*, and the last argument the *verb argument*. Thus, simple quantifiers in **CIT** (m = 0) have only verb arguments. For every $\boldsymbol{Q} \in$ **CIT** and all sets $A_1,\ldots,A_m$, define the simple quantifier

(7) $(\boldsymbol{Q}^{A_1,\ldots,A_m})_M R \quad \Leftrightarrow \quad \boldsymbol{Q}_M A_1\ldots A_m,R$

(if some A_j is not included in M, $(\boldsymbol{Q}^{A_1,\ldots,A_m})_M R$ is false).

Now, the idea is to first define iteration for simple quantifiers, and then extend the definition to all quantifiers in **CIT** via (7). We need the following

1.2. Notation. If R is an n-ary relation on M, $k < n$, and $a_1,\ldots,a_k \in M$, let $R_{a_1\ldots a_k}$ be the $(n-k)$-ary relation defined by

$$R_{a_1\ldots a_k} \quad = \quad \{(a_{k+1},\ldots,a_n) \in M^{n\text{-}k} : \ Ra_1\ldots a_n\}.$$

Note that

$$(R_{a_1\ldots a_k})_{b_1\ldots b_m} \quad = \quad R_{a_1\ldots a_k b_1\ldots b_m}\,.$$

Here is how to iterate two simple quantifiers.

1.3. Definition. If $\boldsymbol{Q}_1$ is of type $<k>$, $\boldsymbol{Q}_2$ of type $<m>$, define $\boldsymbol{Q}_1{\cdot}\boldsymbol{Q}_2$ of type $<k{+}m>$ as follows:

$$\boldsymbol{Q}_1{\cdot}\boldsymbol{Q}_2 R \quad \Leftrightarrow \quad \boldsymbol{Q}_1\{(a_1,\ldots,a_k) : \ \boldsymbol{Q}_2 R_{a_1\ldots a_k}\}$$

(the universe M is suppressed as usual).

We will often omit the '·' and write just $\boldsymbol{Q}_1\boldsymbol{Q}_2$. It is easily verified that the iteration operation is associative:

$$(\boldsymbol{Q}_1\boldsymbol{Q}_2)\boldsymbol{Q}_3 \quad = \quad \boldsymbol{Q}_1(\boldsymbol{Q}_2\boldsymbol{Q}_3).$$

Thus, $\boldsymbol{Q}_1\boldsymbol{Q}_2\boldsymbol{Q}_3$, and in general

$$\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k\ ,$$

is well-defined.

We have defined iteration of simple quantifiers in a purely set-theoretic way. Of course, we could have gone via prefixes instead:

1.4. Fact. *If* $\boldsymbol{Q}_i$ *is of type* $<p_i>$, $\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k$ *is the quantifier defined by the sentence*

$$Q_1x_{11}\ldots x_{1p_1} \ldots Q_kx_{k1}\ldots x_{kp_k} Rx_{11}\ldots x_{kp_k} .$$

Finally, we extend the notion of iteration to arbitrary quantifiers in **CIT**.

1.5. Definition. If $\boldsymbol{Q}_i$ is of type $<1,\ldots,1,p_i>$ with m_i+1 arguments, define the quantifier $\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k$ of type $<1,\ldots,1, \ldots ,1,\ldots,1, p_1+\ldots+p_k>$ (with $m_1+\cdots+m_k+1$ arguments) by

$$\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k A_{11}\ldots A_{km_k},R \iff (\boldsymbol{Q}_1^{A_{11},\ldots,A_{1m_1}}\ldots\boldsymbol{Q}_k^{A_{k1},\ldots,A_{km_k}})R .$$

Thus, the class **CIT** is Closed under ITeration. The reader can check that Definition 1.5 indeed gives the truth conditions we wanted in the examples above. To account for ambiguities we can introduce permutations of iterations:

1.6. Definition. For simple $\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k$, a permutation $i_1,\ldots,i_k$ of $1,\ldots,k$ induces a *permutation* $(\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k)^{(i_1,\ldots,i_k)}$ of $\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k$ as follows: $(\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k)^{(i_1,\ldots,i_k)}$ is the quantifier defined by the sentence

$$Q_{i_1}x_{i_11}\ldots x_{i_1p_{i_1}} \ldots Q_{i_k}x_{i_k1}\ldots x_{i_kp_{i_k}} Rx_{11}\ldots x_{kp_k}.$$

This can be extended to arbitrary $\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k \in$ **CIT** as usual:

$$(\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k)^{(i_1,\ldots,i_k)}A_{11}\ldots A_{km_k},R \iff (\boldsymbol{Q}_1^{A_{11},\ldots,A_{1m_1}}\ldots\boldsymbol{Q}_k^{A_{k1},\ldots,A_{km_k}})^{(i_1,\ldots,i_k)}R.$$ [3]

2 Basic properties of iterations

The familiar properties of type <1,1> quantifiers,

CONSERV	$\boldsymbol{Q}_MAB \iff \boldsymbol{Q}_MA\, A\cap B$
EXT	If $A,B \subseteq M$ and $A,B \subseteq M'$, then $\boldsymbol{Q}_MAB \iff \boldsymbol{Q}_{M'}AB$
ISOM	If $(M,A,B) \cong (M',A',B')$ then $\boldsymbol{Q}_MAB \iff \boldsymbol{Q}_{M'}A'B'$,

can be generalized to quantifiers in **CIT**. This is immediate for EXT and ISOM. For CONSERV, let $\boldsymbol{Q}$ be of type $<1,\ldots,1,k>$ with $m+1$ arguments, and assume $m > 0$ to avoid trivialities. For example, $\boldsymbol{Q}$ could be an iteration

[3]It would have been aesthetically more satisfactory, though slightly longer, to write down the $L(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)$-sentence defining the permutation also in the general case. On the other hand, a purely set-theoretic formulation of Definition 1.6 is of course possible but considerably more involved.

$\boldsymbol{Q}_1 \ldots \boldsymbol{Q}_r$, and then we know precisely which noun arguments are *linked* to which arguments of the k-ary relation. This information is required for CONSERV to make sense: CONSERV says that each relation argument can be restricted to the *union* of those sets which are linked to it:

2.1. Definition. Let $\boldsymbol{Q}$ be as above (so $m \geq k$). For each $m_1,\ldots,m_k > 0$ with $m_1+\cdots+m_k = m$, we say that $\boldsymbol{Q}$ is $(m_1,\ldots,m_k)$-*conservative* if the following holds:

CONSERV $\quad \boldsymbol{Q}A_{11}\ldots A_{1m_1},\ldots,A_{k1}\ldots A_{km_k},R \quad \Leftrightarrow$
$\boldsymbol{Q}A_{11}\ldots A_{1m_1},\ldots,A_{k1}\ldots A_{km_k},((A_{11}\cup\cdots\cup A_{1m_1})\times\cdots\times(A_{k1}\cup\cdots\cup A_{km_k}))\cap R.$

When $k = m = 1$ we have the old notion of CONSERV for type <1,1> quantifiers. When $k = 1$, the above definition coincides with the notion of conservativity for monadic quantifiers proposed in the literature. Quantifiers like ***ED*** above of type <1,1,2> are (1,1)-conservative. In most cases a specific linking of noun arguments to the verb argument is understood; we then drop the prefix and talk about plain conservativity.

2.2. Fact. *Iteration preserves CONSERV, EXT, and ISOM. Specifically, for CONSERV: if* $\boldsymbol{Q}_1$ *is* $(m_1,\ldots,m_k)$*-conservative and* $\boldsymbol{Q}_2$ *is* $(p_1,\ldots,p_n)$*-conservative, then* $\boldsymbol{Q}_1\boldsymbol{Q}_2$ *is* $(m_1,\ldots,m_k,p_1,\ldots,p_n)$*-conservative.*

Proof. Straightforward calculation. $\dashv$

A quantifier $\boldsymbol{Q}$ is said to be *trivial on M*, if $\boldsymbol{Q}_M$ is either the empty or the universal relation between relations on M (of the appropriate type); otherwise $\boldsymbol{Q}$ is *nontrivial on M*. This is a *local* notion of nontriviality. We also need a *global* notion — one not confined to a particular universe:

2.3. Definition. Let $\boldsymbol{Q}$ and $m_1,\ldots,m_k$ be as in Definition 2.1. $\boldsymbol{Q}$ is $(m_1,\ldots,m_k)$-*nontrivial* — but '$(m_1,\ldots,m_k)$' is usually left out — if there are $n_1,\ldots,n_k \geq 0$ such that whenever $A_{11},\ldots,A_{km_k} \subseteq M$ with $|A_{i1}\cup\cdots\cup A_{im_i}| \geq n_i$, $1 \leq i \leq k$, $\boldsymbol{Q}^{A_{11},\ldots,A_{km_k}}$ is nontrivial on M. If $\boldsymbol{Q}$ is simple we require instead that there be an $n \geq 0$ such that $\boldsymbol{Q}$ is nontrivial on M whenever $|M| \geq n$. $n_1,\ldots,n_k$ (n) are called the *triviality bounds* of $\boldsymbol{Q}$. If the condition is not satisfied, $\boldsymbol{Q}$ is *trivial*.

For example, the type <1,1> quantifier ***at least 5*** is nontrivial, with a triviality bound of 5, but the quantifier

$$\boldsymbol{Q}AB \quad \Leftrightarrow \quad |A| \text{ is even and } |A\cap B| \geq 5$$

is trivial. Note that this quantifier is (globally) trivial even though it is (locally) nontrivial *on* every universe with at least 6 elements. This is because of the special role of the noun arguments in Definition 2.3: to be nontrivial, $\boldsymbol{Q}$ has to be nontrivial on all large enough noun arguments and all surrounding universes, as it were, not just on all large enough universes. Such a special role is well motivated at least for quantifiers satisfying CONSERV and EXT; these conditions hold for most non-simple quantifiers in **CIT** related to natural language.

2.4. Triviality Lemma. (i) *For* $\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k \in$ **CIT**: $\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k$ *is trivial* $\Leftrightarrow$ *some* $\boldsymbol{Q}_i$ *is trivial.*
(ii) (Keenan) *For simple* $\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k$: $\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k$ *is trivial on* M $\Leftrightarrow$ *some* $\boldsymbol{Q}_i$ *is trivial on* M.

$\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k$ can of course also be simple in (i), but the restriction to simple quantifiers is necessary in the local version (ii). To see this, consider the iteration ***every·0*** , where ***0*** is the empty quantifier of type <1>. We have ***every·0*** $A,R \;\Leftrightarrow\; A = \varnothing$, so ***every·0*** is in fact nontrivial on every M, although one of its components is trivial on every M.

Proof of Lemma 2.4. We first give Keenan's proof of (ii), and then derive (i) from (ii).
(ii): An immediate induction shows that it is sufficient to consider the case $k = 2$. Let $\boldsymbol{Q}_1$ be of type <m> and $\boldsymbol{Q}_2$ of type <n>. Thus,

$$\boldsymbol{Q}_1\boldsymbol{Q}_2R \quad\Leftrightarrow\quad \boldsymbol{Q}_1\{(a_1,\ldots,a_m) : \boldsymbol{Q}_2R_{a_1\ldots a_m}\}.$$

Now, if either $\boldsymbol{Q}_1$ or $\boldsymbol{Q}_2$ is trivial on M, it is straightforward to calculate that so is $\boldsymbol{Q}_1\boldsymbol{Q}_2$. So suppose $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ are both nontrivial on M. Hence there are $R_1,R_2 \subseteq M^m$ and $S_1,S_2 \subseteq M^n$ such that $\boldsymbol{Q}_1R_1$, $\neg\boldsymbol{Q}_1R_2$, $\boldsymbol{Q}_2S_1$, and $\neg\boldsymbol{Q}_2S_2$ (on M). It follows that the following claim establishes the result, i.e., that $\boldsymbol{Q}_1\boldsymbol{Q}_2$ is nontrivial on M:

CLAIM: $\forall R \subseteq M^m\, \exists R' \subseteq M^{m+n}\, (R = \{(a_1,\ldots,a_m) : \boldsymbol{Q}_2R'_{a_1\ldots a_m}\})$.

The Claim is proved by taking R' =

$$\{(a_1,\ldots,a_m,b_1,\ldots,b_n) \colon (Ra_1\ldots a_m \;\&\; S_1b_1\ldots b_n) \vee (\neg Ra_1\ldots a_m \;\&\; S_2b_1\ldots b_n)\}.$$

Then $Ra_1\ldots a_m$ implies that $R'_{a_1\ldots a_m} = S_1$, and hence $\boldsymbol{Q}_2R'_{a_1\ldots a_m}$. Similarly, $\neg Ra_1\ldots a_m$ implies $\neg\boldsymbol{Q}_2R'_{a_1\ldots a_m}$.
(i): We leave it as an exercise to check that if all of $\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k$ are nontrivial, (ii) can be used to verify that $\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k$ too is nontrivial (with triviality bounds given by those for $\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k$). For the other direction, suppose some $\boldsymbol{Q}_i$ is trivial. More precisely, suppose it is $(m_1,\ldots,m_p)$-trivial, and hence is

of type <1,…,1,*p*> with $m_1+\cdots+m_p+1$ arguments. Now choose any putative triviality bounds $n_1,\ldots,n_r$ for $\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k$. Let $n = max(n_1,\ldots,n_r)$. By the triviality of $\boldsymbol{Q}_i$, we can find M and $A_{11},\ldots,A_{pm_p} \subseteq M$ such that $|A_{j1}\cup\cdots\cup A_{jm_j}| \geq n$ for $1 \leq j \leq p$, and $\boldsymbol{Q}_i^{A_{11},\ldots,A_{pm_p}}$ is trivial on M. Now, from (ii) and Definition 1.5 it follows that for *any* choice of the remaining noun arguments for $\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k$ — let us indicate such a choice by $\underline{C}$ — $(\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k)^{\underline{C},A_{11},\ldots,A_{pm_p}}$ is trivial on M. Moreover, by the choice of n, we can take $\underline{C}$ such that all the sizes of the relevant unions of sets are > $n_1,\ldots,n_r$, and all noun arguments are still subsets of M. So we have shown that however these bounds are chosen, we can find an M including noun arguments 'above' the respective bounds such that the corresponding simple 'instance' of $\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k$ is trivial on M. In other words, $\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k$ is trivial. —|

Next, let us look at iteration and negation. For $\boldsymbol{Q}$ of type <1,…,1,*p*>, the *inner negation* $\boldsymbol{Q}\neg$ of $\boldsymbol{Q}$ is defined by $(\boldsymbol{Q}\neg)_M A_1\ldots A_m,R \Leftrightarrow \boldsymbol{Q}_M A_1\ldots A_m,M^p - R$, and the *dual* is $\boldsymbol{Q}^{\mathrm{d}} = \neg(\boldsymbol{Q}\neg) = (\neg\boldsymbol{Q})\neg$. The following lemma is simple but useful.

2.5. Negation Lemma. *For* $\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k$ □ **CIT**:

(i) $\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k = \boldsymbol{Q}_1\ldots\boldsymbol{Q}_{i-1}\cdot\boldsymbol{Q}_i\neg\cdot\neg\boldsymbol{Q}_{i+1}\cdot\boldsymbol{Q}_{i+2}\ldots\boldsymbol{Q}_k$

(ii) $\neg(\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k) = \neg\boldsymbol{Q}_1\cdot\boldsymbol{Q}_2\ldots\boldsymbol{Q}_k$

(iii) $(\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k)\neg = \boldsymbol{Q}_1\ldots\boldsymbol{Q}_{k-1}\cdot\boldsymbol{Q}_k\neg$

(iv) $(\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k)^{\mathrm{d}} = \boldsymbol{Q}_1^{\mathrm{d}}\ldots\boldsymbol{Q}_k^{\mathrm{d}}$

Proof. Almost immediate, using Fact 1.4, and the fact that $(Q\neg)x_1\ldots x_p\phi \times Qx_1\ldots x_p\neg\phi$. —|

Call a simple quantifier $\boldsymbol{Q}$ *positive* (on M) if $\neg\boldsymbol{Q}\varnothing$ (on M). One frequent use of the Negation Lemma is that when $\boldsymbol{Q}$ is a simple iteration, we can always assume that $\boldsymbol{Q}$ is of the form $\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k$ with $\boldsymbol{Q}_2\ldots\boldsymbol{Q}_k$ positive (on a particular M, or on every M).

Our last two lemmas, which are more or less implicit in Keenan 1992, concern the characteristic behaviour of iterations on Cartesian products. In particular the first of these lemmas turns out to be very useful.

2.6. Product Decomposition Lemma. *Suppose* $\boldsymbol{Q}_1$ *is of type* <*k*>, $\boldsymbol{Q}_2$ *of type* <*m*>, *and that* $\boldsymbol{Q}_2$ *is positive on* M. *Then, for all* $R \subseteq M^k$ *and all* $S \subseteq M^m$,

$$\boldsymbol{Q}_1\boldsymbol{Q}_2 R\times S \quad \Leftrightarrow \quad (\boldsymbol{Q}_1 R \ \& \ \boldsymbol{Q}_2 S) \vee (\boldsymbol{Q}_1\varnothing \ \& \ \neg\boldsymbol{Q}_2 S).$$

Proof. This is almost immediate once you understand the mechanism of iteration. The argument goes like this. Since

(a) $Ra_1\ldots a_k \;\Rightarrow\; (R\times S)_{a_1\ldots a_k} = S$
(b) $\neg Ra_1\ldots a_k \;\Rightarrow\; (R\times S)_{a_1\ldots a_k} = \varnothing$,

it follows from the positivity of $\boldsymbol{Q}_2$ that

(c) $\boldsymbol{Q}_2 S \;\Rightarrow\; \{(a_1,\ldots,a_k) : \boldsymbol{Q}_2(R\times S)_{a_1\ldots a_k}\} = R$
(d) $\neg\boldsymbol{Q}_2 S \;\Rightarrow\; \{(a_1,\ldots,a_k) : \boldsymbol{Q}_2(R\times S)_{a_1\ldots a_k}\} = \varnothing$.

And since $\boldsymbol{Q}_1\boldsymbol{Q}_2 R\times S \;\Leftrightarrow\; \boldsymbol{Q}_1\{(a_1,\ldots,a_k): \boldsymbol{Q}_2(R\times S)_{a_1\ldots a_k}\}$, the desired result follows readily from (c) and (d). —|

2.7. Product Lemma. *Suppose that, on M,* $\boldsymbol{Q} = \boldsymbol{Q}_1\ldots\boldsymbol{Q}_k$, *where* $\boldsymbol{Q}_i$ *is of type* $<p_i>$, *and let* $m = p_1+\cdots+p_k$. *Then, for every* $R \subseteq M^m$, *there is a product* $P = R_1\times\cdots\times R_k$, *with* $R_i \subseteq M^{p_i}$, *such that* $\boldsymbol{Q}_M R \;\Leftrightarrow\; \boldsymbol{Q}_M P$.

Proof. By induction on k, the case $k = 1$ being trivial. Suppose the result holds for k, and consider $\boldsymbol{Q}_0\ldots\boldsymbol{Q}_k$, where we may assume that $\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k$ is positive. Take any $R \subseteq M^{p_0+\ldots+p_k}$. Let

$$R_0 \;=\; \{(a_1,\ldots,a_{p_0}) : \boldsymbol{Q}_1\ldots\boldsymbol{Q}_k R_{a_1\ldots a_{k_0}}\}.$$

Thus, $\boldsymbol{Q}_0\ldots\boldsymbol{Q}_k R \;\Leftrightarrow\; \boldsymbol{Q}_0 R_0$. If $R_0 = \varnothing$, we can take $P = \varnothing$. So suppose $R_0 \neq \varnothing$. Since $\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k R_{b_1\ldots b_{k_0}}$ for some $(b_1,\ldots,b_{p_0})$, there is by induction hypothesis a product $P' = R_1\times\cdots\times R_k$ such that $\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k P'$. Set $P = R_0\times R_1\times\cdots\times R_k$. Essentially the same argument as in the previous proof now gives the result.[4] —|

3. Keenan's Prefix Theorems

The two theorems by Keenan mentioned in the Introduction provide answers to the following questions:

1. To what extent does an iteration determine its components, or, equivalently, the prefix that defines it?
2. Are iterations uniquely determined by their behaviour on Cartesian products?

Versions of these answers are given in this section. To distinguish them from Keenan's original theorems, I will call them the Prefix Theorem and

[4]The Product Lemma is very weak. Call a simple quantifier $\boldsymbol{Q}$ *nontrivial on products on M* if there is a product on M (of the suitable form) for which $\boldsymbol{Q}$ holds, and another for which it does not hold. Then $\boldsymbol{Q}$ trivially has the property stated in the Product Lemma. The Product Lemma just gives a way (for iterations) of finding a corresponding product regardless of such a nontriviality assumption.

the Product Theorem, respectively. They generalize Keenan's results in that they (i) are global, not (only) local, and (ii) apply to iterations of arbitrary quantifiers in **CIT** whereas Keenan deals with iterations of type <1> quantifiers.

Starting with the first question, it is clear by associativity that an iteration by itself determines neither the number of its components nor their types. In other words, the notion of a component is not yet precise enough.

3.1. Definition. If $\tau_1,\ldots,\tau_k$ are types of quantifiers in **CIT**, $\sigma = <\tau_1,\ldots,\tau_k>$ is called a (k-ary) *iteration form*. σ is an iteration form *of* $\boldsymbol{Q}$ if there are $\boldsymbol{Q}_i$ of type τ_i such that $\boldsymbol{Q} = \boldsymbol{Q}_1\ldots\boldsymbol{Q}_k$. $\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k$ are called σ-*components* of $\boldsymbol{Q}$.

A first version of the first question is then: Does an iteration $\boldsymbol{Q}$ of form σ determine its σ-components? The answer is NO. First, if $\boldsymbol{Q}$ is trivial we can, by the Triviality Lemma, get no useful information about the σ-components (except that at least one of them must also be trivial). Second, the Negation Lemma shows that there are 2^{k-1} different ways to distribute inner and outer negations in $\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k$ without changing the resulting quantifier or the iteration form.

Thus, we need to disregard trivial quantifiers and provide some information about how negations are distributed. Moreover, we wish to do this globally, not just on a particular universe. A global notion of nontriviality was introduced in Definition 2.3. As to negations, it turns out that it suffices to know, for each choice of the noun arguments, the behaviour of the σ-components when the verb argument is the empty relation. We could extend iteration forms to, say, *weighted* iteration forms by encoding this information as well. The Prefix Theorem then says that a nontrivial iteration together with a weighted iteration form does determine the components uniquely. This formulation is slightly cumbersome, so we proceed instead as follows.

3.2. Definition. Let $\boldsymbol{Q},\boldsymbol{Q}'$ be nontrivial quantifiers of type $<1,\ldots,1,p>$ with $m+1$ arguments. $\boldsymbol{Q}$ and $\boldsymbol{Q}'$ are *balanced*, if for all large enough M and $A_1,\ldots,A_m \subseteq M$ (i.e., with the cardinality of the each relevant union above the maxima of the corresponding triviality bounds), $\boldsymbol{Q}A_1\ldots A_m,\varnothing \Leftrightarrow \boldsymbol{Q}'A_1\ldots A_m,\varnothing$. For $k \geq 2$, the *sequences* $(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)$ and $(\boldsymbol{Q}_1',\ldots,\boldsymbol{Q}_k')$ are *balanced* if $\boldsymbol{Q}_i$ and $\boldsymbol{Q}_i'$ are balanced for $2 \leq i \leq k$ (we actually don't need to assume that $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_1'$ are balanced!). The corresponding local notion of *balance on* M is obtained by restricting attention to a particular universe M and leaving out the nontriviality requirements.

3.3. The Prefix Theorem. *Suppose* $\boldsymbol{Q}_1 \ldots \boldsymbol{Q}_k = \boldsymbol{Q}_1{}' \ldots \boldsymbol{Q}_k{}'$, *where* $(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)$ and $(\boldsymbol{Q}_1{}',\ldots,\boldsymbol{Q}_k{}')$ *are nontrivial and balanced. Then for each i,* $\boldsymbol{Q}_i$ *is eventually equal to* $\boldsymbol{Q}_i{}'$ *(they are equal above the triviality bounds of* $\boldsymbol{Q}_i$*). For the local version we must assume that the quantifiers involved are all simple; then, if* $\boldsymbol{Q}_1 \ldots \boldsymbol{Q}_k = \boldsymbol{Q}_1{}' \ldots \boldsymbol{Q}_k{}'$ *on M, where* $(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)$ *and* $(\boldsymbol{Q}_1{}',\ldots,\boldsymbol{Q}_k{}')$ *are nontrivial and balanced on M, we have for each i,* $\boldsymbol{Q}_i = \boldsymbol{Q}_i{}'$ *on M.*[5]

Remark. Keenan's Generalized Linear Prefix Theorem is essentially the local version of this for $k = 2$ without the assumption of balance. The conclusion then becomes that *either* $\boldsymbol{Q}_1 = \boldsymbol{Q}_1{}'$ and $\boldsymbol{Q}_2 = \boldsymbol{Q}_2{}'$ on M, *or* $\boldsymbol{Q}_1 = \boldsymbol{Q}_1{}'\neg$ and $\boldsymbol{Q}_2 = \neg\boldsymbol{Q}_2{}'$ on M. Balance reduces the options to one, and hence allows generalization to any k.

The answer to the question whether iterations are determined by their product behaviour is YES, once we make clear what 'product behaviour' means.

3.4. Definition. Two quantifiers $\boldsymbol{Q}$ and $\boldsymbol{Q}'$ in **CIT** of the same iteration form $\sigma = <\tau_1,\ldots,\tau_k>$, where $\tau_i = <1,\ldots,1,p_i>$ with m_i+1 arguments, are said to be *equal on products on M w.r.t.* σ, if for all $A_{11},\ldots,A_{km_k} \subseteq M$ and all $R_i \subseteq M^{p_i}$, $\boldsymbol{Q}A_{11}\ldots A_{km_k},R_1\times\cdots\times R_k \Leftrightarrow \boldsymbol{Q}'A_{11}\ldots A_{km_k},R_1\times\cdots\times R_k$ on M. They are *equal on products w.r.t.* σ if this holds for all M.

3.5. The Product Theorem. *If two iterations in* **CIT** *are equal on products (on M) w.r.t. the same iteration form, then they are equal (on M).*

Proof. First, it is clearly enough to prove the local version. Second, it suffices to prove the result for simple quantifiers. For then, if $\boldsymbol{Q}$ and $\boldsymbol{Q}'$ are arbitrary iterations in **CIT** which are equal on products w.r.t. σ on M, choose noun arguments $A_{11},\ldots,A_{km_k} \subseteq M$. By Definition 1.5, $\boldsymbol{Q}^{A_{11},\ldots,A_{km_k}}$ and $\boldsymbol{Q}'^{A_{11},\ldots,A_{km_k}}$ are simple iterations, equal on products on M w.r.t. the simple iteration form σ' corresponding to σ. Hence $\boldsymbol{Q}^{A_{11},\ldots,A_{km_k}} = \boldsymbol{Q}'^{A_{11},\ldots,A_{km_k}}$ on M, and since $A_{11},\ldots,A_{km_k}$ were arbitrary, $\boldsymbol{Q} = \boldsymbol{Q}'$ on M.

Next, we dispose of the case where one of the iterations is trivial on M. So suppose $\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k$ and $\boldsymbol{Q}_1{}'\ldots\boldsymbol{Q}_k{}'$ are equal on products on M w.r.t. the iteration form $<<p_1>,\ldots,<p_k>>$, and that, say, $\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k$ is trivial on M (the other case is symmetric). Let $p = p_1+\cdots+p_k$. Suppose

[5] I am indebted to Dorit Ben-Shalom for this formulation of the Prefix Theorem. My original formulation used a stronger notion of balance, but she pointed out that the present notion is sufficient.

$\forall R \subseteq M^p \boldsymbol{Q}_1 \ldots \boldsymbol{Q}_k R$.

Then, we claim, the same holds for $\boldsymbol{Q}_1' \ldots \boldsymbol{Q}_k'$. For, it follows from our assumption that $\boldsymbol{Q}_1' \ldots \boldsymbol{Q}_k' P$ for any product $P = R_1 \times \cdots \times R_k \subseteq M^p$. But then, by the Product Lemma, $\boldsymbol{Q}_1' \ldots \boldsymbol{Q}_k' R$ holds for all $R \subseteq M^p$. A similar argument applies if $\forall R \subseteq M^p \neg \boldsymbol{Q}_1 \ldots \boldsymbol{Q}_k R$. Hence, $\boldsymbol{Q}_1 \ldots \boldsymbol{Q}_k = \boldsymbol{Q}_1' \ldots \boldsymbol{Q}_k'$.

To prove the theorem for simple iterations which are nontrivial on M we use induction on the length k of the iteration form. The result is trivial for $k = 1$, so suppose it holds for k, and let $\boldsymbol{Q}_0 \ldots \boldsymbol{Q}_k$ and $\boldsymbol{Q}_0' \ldots \boldsymbol{Q}_k'$ be equal on products on M w.r.t. $<<p_0>,\ldots,<p_k>>$. As noted before, we can assume that $\boldsymbol{Q}_1 \ldots \boldsymbol{Q}_k$ and $\boldsymbol{Q}_1' \ldots \boldsymbol{Q}_k'$ are positive on M. Thus, by product decomposition:

(*) For all $R \subseteq M^{p_0}$ and all $S \subseteq M^p$ ($p = p_1 + \cdots + p_k$),

$$(\boldsymbol{Q}_0 R \;\&\; \boldsymbol{Q}_1 \ldots \boldsymbol{Q}_k S) \vee (\boldsymbol{Q}_0 \varnothing \;\&\; \neg \boldsymbol{Q}_1 \ldots \boldsymbol{Q}_k S)$$
$$\Leftrightarrow \; (\boldsymbol{Q}_0' R \;\&\; \boldsymbol{Q}_1' \ldots \boldsymbol{Q}_k' S) \vee (\boldsymbol{Q}_0' \varnothing \;\&\; \neg \boldsymbol{Q}_1' \ldots \boldsymbol{Q}_k' S).$$

Then,

(i) $\boldsymbol{Q}_0 \varnothing \;\; \Leftrightarrow \;\; \boldsymbol{Q}_0' \varnothing$

(we suppress mention of M here and below). To see this, suppose, say, that $\neg \boldsymbol{Q}_0 \varnothing$ but $\boldsymbol{Q}_0' \varnothing$. But then (*) is false for $R = \varnothing$.

(ii) $\boldsymbol{Q}_1 \ldots \boldsymbol{Q}_k$ and $\boldsymbol{Q}_1' \ldots \boldsymbol{Q}_k'$ are equal on products.

This is proved as follows. Suppose first that $\neg \boldsymbol{Q}_0 \varnothing$, and so $\neg \boldsymbol{Q}_0' \varnothing$ by (i). Take any product $P = R_1 \times \cdots \times R_k$. Fix R such that $\boldsymbol{Q}_0 R$ (nontriviality of $\boldsymbol{Q}_0$). Then

$\boldsymbol{Q}_1 \ldots \boldsymbol{Q}_k P$	$\Rightarrow$	$\boldsymbol{Q}_0 R \;\&\; \boldsymbol{Q}_1 \ldots \boldsymbol{Q}_k P$	
	$\Rightarrow$	$\boldsymbol{Q}_0 \ldots \boldsymbol{Q}_k R \times P$	(product decomposition)
	$\Rightarrow$	$\boldsymbol{Q}_0' \ldots \boldsymbol{Q}_k' R \times P$	(assumption)
	$\Rightarrow$	$\boldsymbol{Q}_1' \ldots \boldsymbol{Q}_k' P$	(product decomposition).

Similarly, $\boldsymbol{Q}_1' \ldots \boldsymbol{Q}_k' P \;\Rightarrow \boldsymbol{Q}_1 \ldots \boldsymbol{Q}_k P$. If instead $\boldsymbol{Q}_0 \varnothing$, and hence $\boldsymbol{Q}_0' \varnothing$, apply the above argument to $\neg \boldsymbol{Q}_0 \ldots \boldsymbol{Q}_k$ and $\neg \boldsymbol{Q}_0' \ldots \boldsymbol{Q}_k'$. This proves (ii).

By (ii) and the induction hypothesis,

(iii) $\boldsymbol{Q}_1 \ldots \boldsymbol{Q}_k = \boldsymbol{Q}_1' \ldots \boldsymbol{Q}_k'$.

Now take S such that $\boldsymbol{Q}_1 \ldots \boldsymbol{Q}_k S$. It follows immediately from (iii) and (*) that

(iv) $\boldsymbol{Q}_0 = \boldsymbol{Q}_0'$.

This concludes the proof. —|

The proof of the Prefix Theorem uses a similar argument. We know that the hypothesis that the two iterations are equal can be replaced by the hypothesis that they are equal on products. Thus, if we make the additional assumptions of nontriviality and balance, it is only natural that the conclusion becomes stronger: not only the iterations are equal, but also their respective components.

Proof of the Prefix Theorem. We first show that the local version implies the global one. So suppose $\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k$ and $\boldsymbol{Q}_1',\ldots,\boldsymbol{Q}_k'$ satisfy the assumptions of the global result. Take large enough M and $A_{11},\ldots,A_{km_k} \subseteq M$ (so that the cardinality of the relevant unions are above the respective triviality bounds for $\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k$). Then the quantifiers $\boldsymbol{Q}_1^{A_{11},\ldots,A_{1m_1}},\ldots,\boldsymbol{Q}_k^{A_{k1},\ldots,A_{km_k}}$ and $\boldsymbol{Q}_1'^{A_{11},\ldots,A_{1m_1}},\ldots,\boldsymbol{Q}_k'^{A_{k1},\ldots,A_{km_k}}$ satisfy the assumptions of the local version, relative to M. In particular, they are nontrivial on M by definition. So the local result gives us that $\boldsymbol{Q}_i^{A_{i1},\ldots,A_{im_i}} = \boldsymbol{Q}_i'^{A_{i1},\ldots,A_{im_i}}$ on M, for each i. Hence $\boldsymbol{Q}_i = \boldsymbol{Q}_i'$, for large enough arguments.

The local result is proved by induction. Suppose the result holds for k, and that we have $\boldsymbol{Q}_1\ldots\boldsymbol{Q}_{k+1} = \boldsymbol{Q}_1'\ldots\boldsymbol{Q}_{k+1}'$ on M, where $\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_{k+1}$ and $\boldsymbol{Q}_1',\ldots,\boldsymbol{Q}_{k+1}'$ are nontrivial and balanced on M. Let $\boldsymbol{Q} = \boldsymbol{Q}_1\ldots\boldsymbol{Q}_k$ and $\boldsymbol{Q}' = \boldsymbol{Q}_1'\ldots\boldsymbol{Q}_k'$, so that

$$\boldsymbol{Q}\boldsymbol{Q}_{k+1} = \boldsymbol{Q}'\boldsymbol{Q}_{k+1}'$$

(suppressing M). Distinguish two cases.

Case I: $\neg\boldsymbol{Q}_{k+1}\varnothing$.

By balance, $\neg\boldsymbol{Q}_{k+1}'\varnothing$. Now we argue as in steps (*), (i), (ii) and (iv) of the preceding proof (with $k = 1$), using our assumption of nontriviality on M, concluding successively that $\boldsymbol{Q}\varnothing \Leftrightarrow \boldsymbol{Q}'\varnothing$ (i.e., $\boldsymbol{Q}$ and $\boldsymbol{Q}'$ are balanced), $\boldsymbol{Q}_{k+1} = \boldsymbol{Q}_{k+1}'$, and $\boldsymbol{Q} = \boldsymbol{Q}'$.

Case II: $\boldsymbol{Q}_{k+1}\varnothing$.

By balance, $\boldsymbol{Q}_{k+1}'\varnothing$. But $\boldsymbol{Q}\neg\cdot\neg\boldsymbol{Q}_{k+1} = \boldsymbol{Q}'\neg\cdot\neg\boldsymbol{Q}_{k+1}'$ by the Negation Lemma, so we can conclude as in Case I that $\neg\boldsymbol{Q}_{k+1} = \neg\boldsymbol{Q}_{k+1}'$ and $\boldsymbol{Q}\neg = \boldsymbol{Q}'\neg$.

Thus, in both cases, $\boldsymbol{Q}_{k+1} = \boldsymbol{Q}_{k+1}'$, and $\boldsymbol{Q} = \boldsymbol{Q}'$on M. It then follows from the induction hypothesis that $\boldsymbol{Q}_i = \boldsymbol{Q}_i'$ on M for $1 \leq i \leq k$, and the proof is complete. —|

II
APPLICATIONS TO DEFINABILITY

4. Convertibility and resumptions

Keenan's motive for studying generalized quantifier prefixes was partly to obtain methods for showing that certain polyadic quantifiers are *not* iterations. One such method is this: Show that a particular quantifier is equal to some iteration on products, but not on all relations. Then it follows by the Product Theorem that the quantifier in question cannot be an iteration. This is quite efficient in many cases; for example, it can be used to show that the quantifiers ***ED*** and ***EH*** from section 1 are not iterations.[6]

In sections 4–6 we apply the results in Part I not to particular quantifiers, but to some natural classes of quantifiers, and give necessary and sufficient conditions for a quantifier in such a class to be an iteration (these characterizations are in fact also quite useful for showing particular quantifiers not to be iterations). Although the characterizations are not simple applications of the previous results but require extra work, the basic facts of Part I are used repeatedly, and the present results would have been much less feasible without them.

We now make a few assumptions that will hold, unless otherwise stated, for the rest of the paper. *First*, we restrict attention to *simple* quantifiers. We have already seen that this is no real restriction — think of these quantifiers as *noun phrase denotations*, obtained from quantifiers in **CIT** by fixing the noun arguments. *Second*, we consider for simplicity only iteration forms <<1>,…,<1>>. So the information about a quantifier that it is a *k*-ary iteration uniquely determines its iteration form (and its type, i.e., <*k*>). *Third*, we restrict attention to *finite* universes. *Fourth* and last, we assume ISOM of all quantifiers.

In contrast with the first three assumptions, the last one may seem completely unrealistic from a linguistic point of view, since a noun phrase denotation of the form $\boldsymbol{Q}^{A_{11},\ldots}$ practically *never* satisfies ISOM ($A_{11},\ldots$ are *fixed* sets)! However, this simplifying assumption does have an adequate motivation: our results in fact have stronger versions which only rely on the (quite realistic) assumption that $\boldsymbol{Q}$, not $\boldsymbol{Q}^{A_{11},\ldots}$, satisfies (CONSERV and) ISOM.

The present section is supposed to deal with convertible quantifiers and with resumptions, so we had better define these notions. First, however, another piece of

[6]More examples can be found in Keenan 1992. Ben-Shalom 1993 further extends Keenan's methods.

4.1. Notation. If R is a k-ary relation on M and $i_1,\ldots,i_k$ a permutation of $1,\ldots,k$, let $R^{(i_1,\ldots,i_k)}$ be the relation on M defined by the following condition:

$$R^{(i_1,\ldots,i_k)}a_{i_1}\ldots a_{i_k} \quad \Leftrightarrow \quad Ra_1\ldots a_k.$$

Thus, if R is binary, $R^{(2,1)}$ is the *converse* of R, or R^{-1}. This notation is related to the one we introduced for permutations of iterations in Definition 1.6 by the following easily verified

4.2. Fact. *For type <1> quantifiers, we have* $(\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k)^{(i_1,\ldots,i_k)}R \Leftrightarrow \boldsymbol{Q}_{i_1}\ldots\boldsymbol{Q}_{i_k}R^{(i_1,\ldots,i_k)}$.

The notion of convertibility generalizes the property of a type <2> quantifier of being 'closed under converses' to arbitrary simple quantifiers.

4.3. Definition. A quantifier $\boldsymbol{Q}$ of type $<k>$ is *convertible* (on M) if for every permutation $i_1,\ldots,i_k$ of $1,\ldots,k$ and every k-ary relation R (on M), $\boldsymbol{Q}R \Rightarrow \boldsymbol{Q}R^{(i_1,\ldots,i_k)}$ (on M).

4.4. Fact. (i) *If* $\boldsymbol{Q}$ *is convertible, so are* $\neg\boldsymbol{Q}$, $\boldsymbol{Q}\neg$, *and (hence)* $\boldsymbol{Q}^{\mathrm{d}}$.
(ii) *If* $\boldsymbol{Q}$ *is of type <k> and closed under permutations of k-tuples, then* $\boldsymbol{Q}$ *is convertible.*

Proof. For (i), note that $M^k - R^{(i_1,\ldots,i_k)} = (M^k - R)^{(i_1,\ldots,i_k)}$. (ii) follows from the fact that if $\boldsymbol{Q}$ is closed under permutations of k-tuples, then only the *cardinality* of R matters for whether $\boldsymbol{Q}R$ holds or not, so $\boldsymbol{Q}$ is clearly convertible. $\dashv$

There are lots of convertible quantifiers, but for our main example we need one more

4.5. Definition. If $\boldsymbol{Q}$ is of type <1,…,1> with n arguments and $k \geq 1$, the *k-ary resumption* $\boldsymbol{Q}^{(k)}$ *of* $\boldsymbol{Q}$, of type $<k,\ldots,k>$, is defined as follows: for all M and all $R_1,\ldots,R_n \subseteq M^k$,

$$\boldsymbol{Q}^{(k)}{}_M R_1\ldots R_n \quad \Leftrightarrow \quad \boldsymbol{Q}_{M^k} R_1\ldots R_n.$$

For $k = 0$ we let $\boldsymbol{Q}^{(0)}{}_M = \mathbf{T}_M$, the trivially true quantifier on M (of the type of $\boldsymbol{Q}$).

For example, $\boldsymbol{most}^{(2)}$ is the type <2,2> quantifier defined by

$$\boldsymbol{most}^{(2)}RS \quad \Leftrightarrow \quad |R\cap S| > |R - S|.$$

The use of resumption (quantification over pairs) in natural language is proposed in May 1989; cf. van Benthem 1989 and Westerståhl 1989 for further discussion. Note that $\boldsymbol{most}^{(2)}$ is not in **CIT**. To remain within **CIT** we will only consider resumptions of type <1> quantifiers here. In particular, instead of $\boldsymbol{most}^{(2)}$ we consider the type <2> quantifier

$$(\boldsymbol{Q}^{\mathrm{R}})^{(2)}{}_M R \quad \Leftrightarrow \quad |R| > |M^2 - R|,$$

i.e., the resumption of the 'type <1> counterpart' $\boldsymbol{Q}^{\mathrm{R}}$ of ***most*** (the notation '$\boldsymbol{Q}^{\mathrm{R}}$' is because this is sometimes called the 'Rescher quantifier').

Here are some familiar examples of convertible quantifiers:

- All resumptions $\boldsymbol{Q}^{(k)}$ (since $\boldsymbol{Q}$ is assumed to satisfy ISOM, $\boldsymbol{Q}^{(k)}$ is closed under permutations of k-tuples).
- $\boldsymbol{Q}^{\mathrm{E}}{}_n R \;\Leftrightarrow\; R$ is an equivalence relation with at least n equivalence classes.
- $\boldsymbol{Tot}R \;\Leftrightarrow\; R$ is a total ordering of the universe.
- $\boldsymbol{Q}^k R \;\Leftrightarrow\;$ there is an infinite set A such that $A\times\cdots\times A \subseteq R$ (a 'Ramsey quantifier'; only interesting on infinite universes, of course).

Which iterations are convertible? Well, clearly $\exists\ldots\exists$ and $\forall\ldots\forall$, but it is not so easy to find other examples. Here, however, is one. Let $\boldsymbol{Q}_{\mathrm{odd}}A \Leftrightarrow |A|$ is odd.

4.6. Fact. *$\boldsymbol{Q}_{\mathrm{odd}}{}^{(k)} = \boldsymbol{Q}_{\mathrm{odd}}\cdot\ldots\cdot\boldsymbol{Q}_{\mathrm{odd}}$ (k components). Hence, the iteration $\boldsymbol{Q}_{\mathrm{odd}}\cdot\ldots\cdot\boldsymbol{Q}_{\mathrm{odd}}$ is convertible.*

Proof. Induction on k. $k = 1$ is trivial. Suppose the result is true for k. Write $It^{(k)}(\boldsymbol{Q}_{\mathrm{odd}})$ for $\boldsymbol{Q}_{\mathrm{odd}}\cdot\ldots\cdot\boldsymbol{Q}_{\mathrm{odd}}$ with k components.

$$\begin{aligned} It^{(k+1)}(\boldsymbol{Q}_{\mathrm{odd}})R \quad &\Leftrightarrow \quad \boldsymbol{Q}_{\mathrm{odd}}\{a\colon\ It^{(k)}(\boldsymbol{Q}_{\mathrm{odd}})R_a\} && \text{(by definition)} \\ &\Leftrightarrow \quad \boldsymbol{Q}_{\mathrm{odd}}\{a\colon\ |R_a| \text{ is odd}\} && \text{(by ind. hypothesis).} \end{aligned}$$

But clearly, for $M = \{a_1,\ldots,a_m\}$,

$$|R| \;=\; |R_{a_1}| + \ldots + |R_{a_m}|.$$

Now it is easy to verify that $|R|$ is odd iff there is an odd number of a_i such that $|R_{a_i}|$ is odd. —|

Having found this somewhat unexpected convertible iteration one may ask if there are others. But the main result of this section says, essentially, that the three we have found are the *only* convertible iterations. For the proof we shall need the following

4.7. Lemma. *If $\boldsymbol{Q}_0$ is nontrivial on M and $\boldsymbol{Q}_0 \ldots \boldsymbol{Q}_k$ is convertible on M, then $\boldsymbol{Q}_1 \ldots \boldsymbol{Q}_k$ is also convertible on M.*

Proof. Case 1: $\neg \boldsymbol{Q}_1 \ldots \boldsymbol{Q}_k \varnothing$.

Subcase 1A: $\neg \boldsymbol{Q}_0 \varnothing$.

Take $A \subseteq M$ such that $\boldsymbol{Q}_0 A$ (nontriviality). By product decomposition, for every $R \subseteq M^k$,

$$\boldsymbol{Q}_0 \ldots \boldsymbol{Q}_k A \times R \quad \Leftrightarrow \quad \boldsymbol{Q}_0 A \ \& \ \boldsymbol{Q}_1 \ldots \boldsymbol{Q}_k R \quad \Leftrightarrow \quad \boldsymbol{Q}_1 \ldots \boldsymbol{Q}_k R,$$

and similarly,

$$\boldsymbol{Q}_0 \ldots \boldsymbol{Q}_k A \times R^{(i_1, \ldots, i_k)} \quad \Leftrightarrow \quad \boldsymbol{Q}_1 \ldots \boldsymbol{Q}_k R^{(i_1, \ldots, i_k)}.$$

But $A \times R^{(i_1, \ldots, i_k)}$ is a 'converse' of $A \times R$, and so, by the convertibility of $\boldsymbol{Q}_0 \ldots \boldsymbol{Q}_k$, $\boldsymbol{Q}_1 \ldots \boldsymbol{Q}_k R \Leftrightarrow \boldsymbol{Q}_1 \ldots \boldsymbol{Q}_k R^{(i_1, \ldots, i_k)}$.

Subcase 1B: $\boldsymbol{Q}_0 \varnothing$.

This time, taking A such that $\neg \boldsymbol{Q}_0 A$, product decomposition gives

$$\boldsymbol{Q}_0 \ldots \boldsymbol{Q}_k A \times R \quad \Leftrightarrow \quad \neg \boldsymbol{Q}_1 \ldots \boldsymbol{Q}_k R$$
$$\boldsymbol{Q}_0 \ldots \boldsymbol{Q}_k A \times R^{(i_1, \ldots, i_k)} \quad \Leftrightarrow \quad \neg \boldsymbol{Q}_1 \ldots \boldsymbol{Q}_k R^{(i_1, \ldots, i_k)},$$

etc.

Case 2: $\boldsymbol{Q}_1 \ldots \boldsymbol{Q}_k \varnothing$.

Consider $\boldsymbol{Q}_0 \neg \cdot \neg (\boldsymbol{Q}_1 \ldots \boldsymbol{Q}_k)$ and repeat Case 1. —|

4.8. Theorem. *$\boldsymbol{Q}_1 \ldots \boldsymbol{Q}_k$ is convertible iff, on each M where $\boldsymbol{Q}_1 \ldots \boldsymbol{Q}_k$ is nontrivial, $\boldsymbol{Q}_1 \ldots \boldsymbol{Q}_k$ is either $\exists \ldots \exists$ or $\forall \ldots \forall$ or $\boldsymbol{Q}_{\mathrm{odd}} \cdot \ldots \cdot \boldsymbol{Q}_{\mathrm{odd}}$ (k components), or one of their negations ($k \geq 2$).*

Proof. 'If': This follows from Fact 4.6, and the fact that if $\boldsymbol{Q}$ is trivial on M, it is convertible on M.

'Only if': We shall prove by induction on k that if $\boldsymbol{Q}_1 \ldots \boldsymbol{Q}_k$ is convertible, and nontrivial on M, and if both $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2 \ldots \boldsymbol{Q}_k$ are positive on M, then $\boldsymbol{Q}_1 \ldots \boldsymbol{Q}_k$ is either $\exists \ldots \exists$ or $\forall \ldots \forall$ or $\boldsymbol{Q}_{\mathrm{odd}} \cdot \ldots \cdot \boldsymbol{Q}_{\mathrm{odd}}$ on M. This is sufficient, for we can always assume that $\boldsymbol{Q}_2 \ldots \boldsymbol{Q}_k$ is positive on M, and if $\boldsymbol{Q}_1$ is not positive on M, we apply the result to $\neg \boldsymbol{Q}_1 \ldots \boldsymbol{Q}_k$, and conclude that $\boldsymbol{Q}_1 \ldots \boldsymbol{Q}_k$ is the negation of one the three alternatives.

INDUCTION BASE, $k = 2$: In this inductive proof the basis case requires more work than the induction step. Suppose $\boldsymbol{Q}_1\boldsymbol{Q}_2$ is convertible and non-trivial on M, and that $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ are positive (from now on in the proof, the phrase 'on M' will usually be omitted). We start with the

CLAIM: $\boldsymbol{Q}_1 = \boldsymbol{Q}_2$

To prove this, we argue as follows. For all $A, B \subseteq M$,

$\boldsymbol{Q}_1\boldsymbol{Q}_2 A{\times}B$	$\Leftrightarrow$	$\boldsymbol{Q}_1\boldsymbol{Q}_2 B{\times}A$	(by convertibility)
	$\Leftrightarrow$	$\boldsymbol{Q}_1 B$ & $\boldsymbol{Q}_2 A$	(by product decomposition, since $\boldsymbol{Q}_1, \boldsymbol{Q}_2$ are positive)
	$\Leftrightarrow$	$\boldsymbol{Q}_2\boldsymbol{Q}_1 A{\times}B$	(product decomposition again).

This shows that $\boldsymbol{Q}_1\boldsymbol{Q}_2$ and $\boldsymbol{Q}_2\boldsymbol{Q}_1$ are equal on products. By the Product Theorem, they are equal. Hence, by the Prefix Theorem ($\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ are balanced since they are both positive), $\boldsymbol{Q}_1 = \boldsymbol{Q}_2$.

Let $M = \{a_1,\ldots,a_m\}$. Because of ISOM (and the fact that M is finite), we can regard $\boldsymbol{Q}_1$ simply as a subset of $\{0,\ldots,m\}$. Thus, we often write $k \in \boldsymbol{Q}_1$ instead of 'there is $A \subseteq M$ with $|A| = k$ such that $\boldsymbol{Q}_1 A$'. We now consider two cases.

Case 1: $1 \notin \boldsymbol{Q}_1$.

Since $0,1 \notin \boldsymbol{Q}_1$, we have $m > 1$ by the nontriviality of $\boldsymbol{Q}_1$. Thus, $\boldsymbol{Q}_1$ cannot be $\exists$ or $\boldsymbol{Q}_{\text{odd}}$. We show that it must be $\forall$. Let n be the smallest number in $\boldsymbol{Q}_1$; n exists by nontriviality, and is > 1 by assumption. It suffices to show that $n = m$. Suppose, for contradiction, that $n < m$. Choose a binary relation R such that a_i has exactly n (R-)successors for $1 \le i \le n$, but such that *not* all of these a_i have the *same* successors, and that the remaining elements of M have no successors. Note that this choice of R is possible because $1 < n < m$. Since $n \in \boldsymbol{Q}_1$ but $0 \notin \boldsymbol{Q}_1$, $|\{a : \boldsymbol{Q}_1 R_a\}| = n$, and so $\boldsymbol{Q}_1\boldsymbol{Q}_1 R$. But, by the construction of R, the number of elements in M with exactly n *predecessors* must be smaller than n (all predecessors are among $a_1,\ldots,a_n$). So this number is *not* in $\boldsymbol{Q}_1$, and it follows that $\neg\boldsymbol{Q}_1\boldsymbol{Q}_1 R^{-1}$. This contradicts the convertibility of $\boldsymbol{Q}_1\boldsymbol{Q}_1$, and we have shown that $\boldsymbol{Q}_1 = \forall = \{m\}$.

Case 2: $1 \in \boldsymbol{Q}_1$.

Subcase 2A: $2 \in \boldsymbol{Q}_1$.

We show that $\boldsymbol{Q}_1 = \exists = \{1,\ldots,m\}$. This follows from the next

CLAIM: If $k \in \boldsymbol{Q}_1$, then $k{+}1 \in \boldsymbol{Q}_1$ ($k < m$).

If $m = 2$ we are done, so assume $m > 2$. Assume $k \in \boldsymbol{Q}_1$ but $k{+}1 \notin \boldsymbol{Q}_1$. Then define R as follows:

$$R_{a_1} = \{a_1,\ldots,a_{k+1}\}$$
$$R_{a_2} = \{a_1,a_2\}$$
$$R_{a_i} = \{a_i\}\text{ , for } 3 \le i \le k{+}1,\text{ and } R_a = \varnothing \text{ otherwise.}$$

Then we have $\{a : \boldsymbol{Q}_1 R_a\} = \{a_2,\ldots,a_{k+1}\}$, so $\boldsymbol{Q}_1\boldsymbol{Q}_1 R$. But $\{a : \boldsymbol{Q}_1 R^{-1}{}_a\} = \{a_1,\ldots,a_{k+1}\}$, and hence $\neg\boldsymbol{Q}_1\boldsymbol{Q}_1 R^{-1}$, contradicting our hypothesis, and the Claim is proved.

Subcase 2B: $2 \notin \boldsymbol{Q}_1$.

That $\boldsymbol{Q}_1 = \boldsymbol{Q}_{\text{odd}}$ now follows from the

CLAIM: $k \notin \boldsymbol{Q}_1$ iff $k{+}1 \in \boldsymbol{Q}_1$ $(k < m)$.

To prove this we define R as follows:

$$R_{a_1} = \{a_1,\ldots,a_k\}$$
$$R_{a_2} = \{a_{k+1}\}\text{ , and } R_a = \varnothing \text{ otherwise.}$$

Suppose first that $k \notin \boldsymbol{Q}_1$ and $k{+}1 \notin \boldsymbol{Q}_1$. Then $\{a : \boldsymbol{Q}_1 R_a\} = \{a_2\}$, so $\boldsymbol{Q}_1\boldsymbol{Q}_1 R$. But $\{a : \boldsymbol{Q}_1 R^{-1}{}_a\} = \{a_1,\ldots,a_{k+1}\}$, and hence $\neg\boldsymbol{Q}_1\boldsymbol{Q}_1 R^{-1}$, contradiction. Next, suppose instead $k \in \boldsymbol{Q}_1$ and $k{+}1 \in \boldsymbol{Q}_1$. This time, $\{a : \boldsymbol{Q}_1 R_a\} = \{a_1,a_2\}$, and so $\neg\boldsymbol{Q}_1\boldsymbol{Q}_1 R$. But $\{a : \boldsymbol{Q}_1 R^{-1}{}_a\} = \{a_1,\ldots,a_{k+1}\}$ as before, and it follows that $\boldsymbol{Q}_1\boldsymbol{Q}_1 R^{-1}$, again a contradiction. This proves the Claim, and thereby concludes the proof of the induction basis.

INDUCTION STEP: Suppose the result is true for k, and let $\boldsymbol{Q}_0\ldots\boldsymbol{Q}_k$ be convertible, nontrivial on M, and such that both $\boldsymbol{Q}_0$ and $\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k$ are positive. Since $\boldsymbol{Q}_0$ is nontrivial (by the Triviality Lemma), it follows from Lemma 4.7 and the induction hypothesis that $\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k$ is either $\exists\ldots\exists$ or $\forall\ldots\forall$ or $\boldsymbol{Q}_{\text{odd}}\cdot\ldots\cdot\boldsymbol{Q}_{\text{odd}}$.

Case 1: $\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k = \exists\ldots\exists$.

Thus, $\boldsymbol{Q}_0\ldots\boldsymbol{Q}_k R \Leftrightarrow \boldsymbol{Q}_0\{a : R_a \neq \varnothing\}$. Since $\boldsymbol{Q}_0$ is positive and nontrivial, there is $A \neq \varnothing$ such that $\boldsymbol{Q}_0 A$. Fix a non-empty $S \subseteq M^{k-1}$, and let B be any non-empty subset of M. Then

$$\boldsymbol{Q}_0\ldots\boldsymbol{Q}_k A{\times}B{\times}S \quad\Leftrightarrow\quad \boldsymbol{Q}_0 A$$
$$\boldsymbol{Q}_0\ldots\boldsymbol{Q}_k B{\times}A{\times}S \quad\Leftrightarrow\quad \boldsymbol{Q}_0 B.$$

Thus, by the convertibility of $\boldsymbol{Q}_0\ldots\boldsymbol{Q}_k$, $\boldsymbol{Q}_0 B$. We have shown that $\boldsymbol{Q}_0 B$ for any non-empty subset B of M, i.e., $\boldsymbol{Q}_0 = \exists$ on M.

Case 2: $\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k = \forall\ldots\forall$.

By the Negation Lemma, $(\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k)^{\mathrm{d}} = \exists\ldots\exists$. But $(\boldsymbol{Q}_0\ldots\boldsymbol{Q}_k)^{\mathrm{d}} = \boldsymbol{Q}_0{}^{\mathrm{d}}\cdot\boldsymbol{Q}_1{}^{\mathrm{d}}\ldots\boldsymbol{Q}_k{}^{\mathrm{d}}$ is convertible (Fact 4.4), and $\boldsymbol{Q}_0{}^{\mathrm{d}}$ and $(\boldsymbol{Q}_1{}^{\mathrm{d}}\ldots\boldsymbol{Q}_k{}^{\mathrm{d}})$ are positive, so by Case 1 we get $\boldsymbol{Q}_0{}^{\mathrm{d}} = \exists$, i.e., $\boldsymbol{Q}_0 = \forall$.

Case 3: $\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k = \boldsymbol{Q}_{\mathrm{odd}}\cdot\ldots\cdot\boldsymbol{Q}_{\mathrm{odd}}$.

This time, by Fact 4.6, $\boldsymbol{Q}_0\ldots\boldsymbol{Q}_k R \Leftrightarrow \boldsymbol{Q}_0\{a : |R_a| \text{ is odd}\}$. Fix $S \subseteq M^{k-1}$ such that $|S|$ is odd. Then, for all $A, B \subseteq M$:

(i) If $|B|$ is odd, $\boldsymbol{Q}_0\ldots\boldsymbol{Q}_k A{\times}B{\times}S \Leftrightarrow \boldsymbol{Q}_0 A$ (since $|B{\times}S|$ is odd).
(ii) If $|A|$ is odd, $\boldsymbol{Q}_0\ldots\boldsymbol{Q}_k B{\times}A{\times}S \Leftrightarrow \boldsymbol{Q}_0 B$.
(iii) If $|A|$ is even, $\neg\boldsymbol{Q}_0\ldots\boldsymbol{Q}_k B{\times}A{\times}S$ (since $|A{\times}S|$ is even and $\neg\boldsymbol{Q}_0\varnothing$).
(iv) If $|A|$ is even, $\neg\boldsymbol{Q}_0 A$ (by convertibility from (iii) and (i)).
(v) If $|B|$ is odd, then $\boldsymbol{Q}_0 B$.

To see that (v) holds, suppose $|B|$ is odd, and take (by nontriviality) an A such that $\boldsymbol{Q}_0 A$. By (iv), $|A|$ is odd. Thus, by (i), (ii), and convertibility, $\boldsymbol{Q}_0 B$.

(iv) and (v) show that $\boldsymbol{Q}_0 = \boldsymbol{Q}_{\mathrm{odd}}$ on M. This concludes the proof of the induction step, and thereby of the theorem. —|

As a bonus, we get the following characterization for free:

4.9. Corollary. *$\boldsymbol{Q}^{(k)}$ is a k-ary iteration iff, on each M where $\boldsymbol{Q}^{(k)}$ is nontrivial, $\boldsymbol{Q}^{(k)}$ is either $\exists^{(k)}$ or $\forall^{(k)}$ or $\boldsymbol{Q}_{\mathrm{odd}}{}^{(k)}$, or one of their negations* ($k \geq 2$).

Proof. 'Only if': If $\boldsymbol{Q}^{(k)} = \boldsymbol{Q}_1\ldots\boldsymbol{Q}_k$, then $\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k$ is convertible, so the result is immediate from the theorem (and Fact 4.6).

'If': Suppose the right hand side holds. We must find type <1> quantifiers $\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k$ such that $\boldsymbol{Q}^{(k)} = \boldsymbol{Q}_1\ldots\boldsymbol{Q}_k$. So we must define each $\boldsymbol{Q}_i$ on every universe M. If $\boldsymbol{Q}^{(k)}$ is trivial on M, we can clearly find $\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k$ on M such that $\boldsymbol{Q}^{(k)} = \boldsymbol{Q}_1\ldots\boldsymbol{Q}_k$ on M. If $\boldsymbol{Q}^{(k)}$ is nontrivial on M, it is either $\exists^{(k)}$ or $\forall^{(k)}$ or $\boldsymbol{Q}_{\mathrm{odd}}{}^{(k)}$, or one of their negations, on M. So again $\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k$ can be defined on M in the desired way. (For example, if $\boldsymbol{Q}^{(k)} = \neg\exists^{(k)}$ on M, we let $(\boldsymbol{Q}_1)_M = \neg\exists_M$, and $(\boldsymbol{Q}_2)_M = \ldots = (\boldsymbol{Q}_k)_M = \exists_M$. Of course, we cannot guarantee that $\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k$ are defined in the *same* way on every M where $\boldsymbol{Q}^{(k)}$ is nontrivial, but that is not required.) —|

Remark: In view of the fact (4.4) that not only $\neg\boldsymbol{Q}$, but also $\boldsymbol{Q}\neg$ and $\boldsymbol{Q}^{\mathrm{d}}$ are convertible if $\boldsymbol{Q}$ is, didn't we forget a few cases in Theorem 4.8? Well, $(\exists\ldots\exists)\neg$ is convertible, but $(\exists\ldots\exists)\neg = \neg(\forall\ldots\forall)$, so this is covered by the theorem. But what about $(\boldsymbol{Q}_{\mathrm{odd}}\cdot\ldots\cdot\boldsymbol{Q}_{\mathrm{odd}})\neg$? To see that this too is covered, note first that

(1) If $|M|$ is odd then $\boldsymbol{Q}_{\mathrm{odd}}{}^{\mathrm{d}} = \boldsymbol{Q}_{\mathrm{odd}}$ on M, and if $|M|$ is even,

$\boldsymbol{Q}_{\text{odd}}\neg = \boldsymbol{Q}_{\text{odd}}$ on M.

Thus, if $|M|$ is odd, then $(\boldsymbol{Q}_{\text{odd}}\cdot\ldots\cdot\boldsymbol{Q}_{\text{odd}})\neg = \neg(\boldsymbol{Q}_{\text{odd}}{}^{\text{d}}\cdot\ldots\cdot\boldsymbol{Q}_{\text{odd}}{}^{\text{d}}) = \neg(\boldsymbol{Q}_{\text{odd}}\cdot\ldots\cdot\boldsymbol{Q}_{\text{odd}})$ as above, and if $|M|$ is even, then $(\boldsymbol{Q}_{\text{odd}}\cdot\ldots\cdot\boldsymbol{Q}_{\text{odd}})\neg = \boldsymbol{Q}_{\text{odd}}\cdot\ldots\cdot\boldsymbol{Q}_{\text{odd}}\cdot\boldsymbol{Q}_{\text{odd}}\neg = \boldsymbol{Q}_{\text{odd}}\cdot\ldots\cdot\boldsymbol{Q}_{\text{odd}}$. —|

It should be noted that both the assumptions of ISOM and of finiteness of the universe are used essentially in these results. As to ISOM, consider typical non-ISOM quantifiers like *proper names*:

$$\boldsymbol{John}A \iff John \in A.$$

Then $\boldsymbol{John}\cdot\boldsymbol{John}R \iff (John,John) \in R$, so ***John·John*** is convertible. Similarly, it can be seen that the results in the next two sections fail for proper names.

However, we said earlier that the requirement of ISOM for the type <1> quantifiers $\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k$ in Theorem 4.8 can be weakened to a linguistically more realistic assumption. To this end we need two more lemmas. Let $\boldsymbol{odd}AB \iff |A\cap B|$ is odd, and recall that for $\boldsymbol{Q}$ of type <1,1>, $(\boldsymbol{Q}^A)_M B \iff \boldsymbol{Q}_M AB$.

4.10. Lemma. *$\boldsymbol{some}^A\cdot\boldsymbol{some}^B$ is convertible iff $A = B$ or one of A,B is $\varnothing$. Similarly for $\boldsymbol{all}^A\cdot\boldsymbol{all}^B$ and $\boldsymbol{odd}^A\cdot\boldsymbol{odd}^B$.*

Proof. If $A = \varnothing$ or $B = \varnothing$, $\boldsymbol{some}^A\cdot\boldsymbol{some}^B R$ is always false, so convertibility holds. And clearly $\boldsymbol{some}^A\cdot\boldsymbol{some}^A$ is convertible. For the other direction, suppose $\varnothing \neq A,B$, and take $a \in A$. Then take $b \in B$ and let $R = \{(a,b)\}$. Thus $\boldsymbol{some}^A\cdot\boldsymbol{some}^B R$, so by convertibility, $\boldsymbol{some}^A\cdot\boldsymbol{some}^B R^{-1}$, i.e., there is $c \in A$ such that $\boldsymbol{some}^B(R^{-1})_c$. Hence, $c = b$, $(R^{-1})_b = \{a\}$, and $B\cap(R^{-1})_b \neq \varnothing$, so $a \in B$. We have shown that $A \subseteq B$, and by symmetry, that $B \subseteq A$. This takes care of the case of $\boldsymbol{some}^A\cdot\boldsymbol{some}^B$. For $\boldsymbol{all}^A\cdot\boldsymbol{all}^B$ the result follows by taking duals. For $\boldsymbol{odd}^A\cdot\boldsymbol{odd}^B$ essentially the same argument as for $\boldsymbol{some}^A\cdot\boldsymbol{some}^B$ works. —|

4.11. Lemma. *Suppose that $\boldsymbol{Q}^A = \boldsymbol{some}^B$ on M, where $\boldsymbol{Q}^A$ and $\boldsymbol{some}^B$ are nontrivial on M, $\boldsymbol{Q}$ is ISOM and CONSERV, and $\neg\boldsymbol{Q}^A\varnothing$ on M. Then $A = B$. The same conclusion holds if $\boldsymbol{Q}^A = \boldsymbol{all}^B$, or $\boldsymbol{Q}^A = \boldsymbol{odd}^B$.*

Proof. Note that the assumptions imply that $\varnothing \neq A,B \subseteq M$ (in particular, if $A = \varnothing$, then by CONSERV, $\boldsymbol{Q}^A C \iff \boldsymbol{Q}^A\varnothing$ for all $C \subseteq M$, which makes $\boldsymbol{Q}^A$ trivial on M). Suppose first $a \in B$. Then $\boldsymbol{some}^B\{a\}$, so $\boldsymbol{Q}^A\{a\}$, and $\boldsymbol{Q}^A A\cap\{a\}$ by CONSERV. Hence, $A\cap\{a\} \neq \varnothing$, i.e., $a \in A$. So $B \subseteq A$. Next, suppose $a \in A$. Take any $b \in B$. Since $\boldsymbol{some}^B\{b\}$, we have $\boldsymbol{Q}^A\{b\}$. But $a,b \in A$, and hence it follows from ISOM that $\boldsymbol{Q}^A\{a\}$. Hence, $\boldsymbol{some}^B\{a\}$, so $a \in$

B. This shows that $A = B$ when $\boldsymbol{Q}^A = \boldsymbol{some}^B$. If instead $\boldsymbol{Q}^A = \boldsymbol{all}^B$, we get the same result by taking duals, and if $\boldsymbol{Q}^A = \boldsymbol{odd}^B$, the same proof as for ***some*** works. —|

We can now give the following strengthening of Theorem 4.8.

4.12. Theorem. *Suppose that the type <1,1> quantifiers* $\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k$ *are CONSERV and ISOM, that* $A_1,\ldots,A_k \neq \emptyset$, *and that* $\boldsymbol{Q}_1^{A_1}\ldots\boldsymbol{Q}_k^{A_k}$ *is nontrivial on some universe. Then,* $\boldsymbol{Q}_1^{A_1}\ldots\boldsymbol{Q}_k^{A_k}$ *is convertible iff* $A_1 = \ldots = A_k = A$, *and, on every* M *where* $\boldsymbol{Q}_1^{A_1}\ldots\boldsymbol{Q}_k^{A_k}$ *is nontrivial,* $\boldsymbol{Q}_1^{A_1}\ldots\boldsymbol{Q}_k^{A_k}$ *is either* $\boldsymbol{some}^A\ldots\boldsymbol{some}^A$ *or* $\boldsymbol{all}^A\ldots\boldsymbol{all}^A$ *or* $\boldsymbol{odd}^A\ldots\boldsymbol{odd}^A$ *(k components), or a negation of these.*

Note that Theorem 4.8 follows from this theorem: if $\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k$ are of type <1> and ISOM, then $\boldsymbol{Q}_1',\ldots,\boldsymbol{Q}_k'$ defined by $\boldsymbol{Q}_i'BC \Leftrightarrow \boldsymbol{Q}_i\, B\cap C$ are CONSERV and ISOM, and $\boldsymbol{Q}_i = (\boldsymbol{Q}_i')^M$ on every M, so the theorem applies.

Proof of Theorem 4.12 (outline). 'If': This is just as before.
'Only if': We consider only the case $k = 2$. Suppose $\boldsymbol{Q}_1^A\boldsymbol{Q}_2^B$ is convertible, and nontrivial on M; such an M exists by hypothesis. As before we may assume that $\boldsymbol{Q}_1^A$ and $\boldsymbol{Q}_2^B$ are both positive on M. Now the first claim in the proof of Theorem 4.8 used only the Product Decomposition Lemma, not ISOM, so exactly the same argument gives

$$\boldsymbol{Q}_1^A = \boldsymbol{Q}_2^B$$

(on M). Thus, $\boldsymbol{Q}_1^A\boldsymbol{Q}_1^A$ is convertible. Since $\boldsymbol{Q}_1$ is CONSERV, only the behaviour of $\boldsymbol{Q}_1^A$ on subsets of A need be considered. Since $\boldsymbol{Q}_1$ is ISOM (and A is finite), only the size of these subsets matters. Thus, $\boldsymbol{Q}_1^A$ can be considered as a subset of $\{0,\ldots,|A|\}$, where $k \in \boldsymbol{Q}_1^A$ means that there is $C \subseteq A$ such that $|C| = k$ and $\boldsymbol{Q}_1^A C$. But then, the same arguments as for the case $k = 2$ in the proof of Theorem 4.8 show that $\boldsymbol{Q}_1^A$ is either $\boldsymbol{some}^A$ or $\boldsymbol{all}^A$ or $\boldsymbol{odd}^A$ on M. It also follows by Lemma 4.11 that $A = B$, and we are done. —|

The results in the next two sections also have stronger versions, where the type <1> quantifiers involved are 'instances' of CONSERV and ISOM type <1,1> quantifiers, but I will not state these versions explicitly.

5. Branching

Barwise 1979 introduced branching generalized quantifiers in connection with natural language semantics. Here we shall only consider branching of MON↑ type <1> quantifiers, defined (by Barwise) as follows:

5.1. Definition. For type <1> MON↑ $\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k$, define the quantifier $B(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)$ of type <k> by

$$B(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)_M R \quad \Leftrightarrow$$
$$\exists X_1,\ldots,X_k \subseteq M\ [\boldsymbol{Q}_{1M}X_1\ \&\ \ldots\ \&\ \boldsymbol{Q}_{kM}X_k\ \ \&\ \ X_1\times\cdots\times X_k \subseteq R].$$

We call the corresponding syntactic expression '$B(Q_1,\ldots,Q_k)$' a *branching prefix* (for typographical reasons; a vertical alignment of $Q_1,\ldots,Q_k$ would have been better).

Note that $B(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)$, as defined above, is always MON↑, regardless of the monotonicity behaviour of $\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k$. However, in what follows we presuppose that $\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k$ are MON↑, whenever '$B(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)$' occurs.

5.2. Lemma. *If each of* $\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k$ *is MON↑, then so is* $\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k$.

Proof. Straightforward calculation. —|

Here are some useful facts about branching.

5.3. Lemma. $B(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)$ *is nontrivial on M iff each* $\boldsymbol{Q}_i$ *is nontrivial on M.*

Proof. 'Only if': Suppose that $\boldsymbol{Q}_i$ is trivial on M. If $\boldsymbol{Q}_i = \varnothing$ then $B(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)R$ is always false. If $\boldsymbol{Q}_i = P(M)$, then $\boldsymbol{Q}_i\varnothing$, and so, by the definition of branching,

$$B(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)R \quad \Leftrightarrow \quad \exists X_1,\ldots,X_k\ [\boldsymbol{Q}_1X_1\ \&\ \ldots\ \&\ \boldsymbol{Q}_kX_k],$$

and the right hand side is independent of R.

'If': Suppose that each $\boldsymbol{Q}_i$ is nontrivial on M. Then there are X_i such that $\boldsymbol{Q}_i X_i$, and hence $B(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)X_1\times\cdots\times X_k$. It remains to show that *not* $\forall R \subseteq M^k$ $B(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)R$. Suppose this is not so. Then $B(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)\varnothing$, so it follows that for some i, $\boldsymbol{Q}_i\varnothing$. But then $\boldsymbol{Q}_i$ is trivial on M by MON↑, contrary to assumption. —|

5.4. Lemma. *If* $B(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)$ *is nontrivial on M, then* $B(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)$ *and* $\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k$ *are equal on products on M.*

Proof. By the previous lemma, each $\boldsymbol{Q}_i$ is non-trivial on M. We have

$$
\begin{array}{lll}
& B(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)A_1\times\cdots\times A_k & \\
\Leftrightarrow & \exists X_1,\ldots,X_k\,[\boldsymbol{Q}_1X_1\ \&\ \ldots\ \&\ \boldsymbol{Q}_kX_k\ \ \&\ \ X_1\times\cdots\times X_k \subseteq A_1\times\cdots\times A_k] & \\
\Leftrightarrow & \exists X_1[\boldsymbol{Q}_1X_1\ \&\ X_1 \prod A_1]\ \&\ \ldots\ \&\ \exists X_k[\boldsymbol{Q}_kX_k\ \&\ X_k \subseteq A_k] & \\
& & (X_i \neq \varnothing \text{ by MON}\uparrow \text{ and nontriviality}) \\
\Leftrightarrow & \boldsymbol{Q}_1A_1\ \&\ \ldots\ \&\ \boldsymbol{Q}_kA_k & (\text{by MON}\uparrow) \\
\Leftrightarrow & \boldsymbol{Q}_1\ldots\boldsymbol{Q}_kA_1\times\cdots\times A_k & (\text{by product decomposition}).
\end{array}
$$

—|

The next lemma shows that the branching of $\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k$ is at least as strong as than the iteration of $\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k$, in any order.

5.5. Lemma. *For any permutation* $i_1,\ldots,i_k$ *of* $1,\ldots,k$ *and any k-ary R,*

$$B(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)R \quad \Rightarrow \quad (\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k)^{(i_1,\ldots,i_k)}\,R.$$

Proof. If there are $X_1,\ldots,X_k$ such that $\boldsymbol{Q}_jX_j$ for each j and $X_1\times\cdots\times X_k \subseteq R$, it follows that

$$(*) \qquad X_{i_1}\times\cdots\times X_{i_k} \ \subseteq\ R^{(i_1,\ldots,i_k)}.$$

Suppose first that $\boldsymbol{Q}_j\varnothing$ for some j. Since $\boldsymbol{Q}_j$ is MON↑, $\{a : \boldsymbol{Q}_j\, A\} = M$ for every set A. But, by MON↑ and our assumption, $\boldsymbol{Q}_iM$ for each i. From this it follows that $(\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k)^{(i_1,\ldots,i_k)}R$.

Next, suppose $\neg\boldsymbol{Q}_j\varnothing$ for each j. Then, by the Product Decomposition Lemma,

$$\boldsymbol{Q}_{i_1}\ldots\boldsymbol{Q}_{i_k}(X_{i_1}\times\cdots\times X_{i_k}).$$

Thus, by (*) and Lemma 5.2, $\boldsymbol{Q}_{i_1}\ldots\boldsymbol{Q}_{i_k}R^{(i_1,\ldots,i_k)}$, i.e., $(\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k)^{(i_1,\ldots,i_k)}R$. —|

This is a good time to state the following fact, which is immediate from the definition of branching:

5.6. Fact. $B(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)R \quad \Leftrightarrow \quad B(\boldsymbol{Q}_{i_1},\ldots,\boldsymbol{Q}_{i_k})R^{(i_1,\ldots,i_k)}$.

We may express this by saying that branching prefixes are *order independent*, in contrast with (most) linear prefixes. Note carefully that order independence, i.e., invariance under permutations of quantifier expressions in a prefix, is a property of syntactic prefixes, but *not* of the quantifiers defined by these prefixes. For example, the quantifiers ∃∃ and

$\exists\neg\cdot\neg\exists$ are identical, but whereas the prefix $\exists x \exists y$ is order independent, the prefix $(\exists\neg)x(\neg\exists)y$ is not.[7]

Also note the difference between order independence of a prefix and convertibility of the corresponding quantifier, i.e., invariance under permutations of the arguments of the relation (this *is* a property of quantifiers). Branching quantifiers are not in general convertible, not even when they are equal to iterations, as we shall see. However, for both linear and branching prefixes, the two properties coincide in the case of iterations of the *same* quantifier expression (compare the notion of self-commutativity in van Benthem 1989), so, for example, $B(\boldsymbol{Q},\ldots,\boldsymbol{Q})$ is convertible. Is this the *only* case when a branching quantifier is convertible? The positive answer to this question (which was posed by Jaap van der Does) turns out to be a simple application of the methods developed here:

5.7. Proposition. *$B(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)$ is convertible iff, on any M where $B(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)$ is nontrivial, $\boldsymbol{Q}_1 = \ldots = \boldsymbol{Q}_k$.*

Proof. Suppose $B(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)$ is convertible and nontrivial on M. Let $i_1,\ldots,i_k$ be any permutation of $1,\ldots,k$, and let $j_1,\ldots,j_k$ be its inverse permutation. Then

$$
\begin{aligned}
B(\boldsymbol{Q}_{i_1},\ldots,\boldsymbol{Q}_{i_k})R \quad &\Leftrightarrow \quad B(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)R^{(j_1,\ldots,j_k)} && \text{(by Fact 5.6)} \\
&\Leftrightarrow \quad B(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)R && \text{(by convertibility).}
\end{aligned}
$$

Thus, $B(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k) = B(\boldsymbol{Q}_{i_1},\ldots,\boldsymbol{Q}_{i_k})$. But then, by Lemmas 5.3 and 5.4, $\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k$ and $\boldsymbol{Q}_{i_1}\ldots\boldsymbol{Q}_{i_k}$ are equal on products (on M). Since each $\boldsymbol{Q}_i$ is positive on M, it follows as usual from the Product and Prefix Theorems that $\boldsymbol{Q}_r = \boldsymbol{Q}_{i_r}$, for $r = 1,\ldots,k$. But since $i_1,\ldots,i_k$ was an arbitrary permutation, this can only hold if $\boldsymbol{Q}_1 = \ldots = \boldsymbol{Q}_k$ on M. $\dashv$

The main result of this section says that branchings are iterations only in very few cases.

5.8. Theorem. *$B(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)$ is a k-ary iteration iff, on each M where $B(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)$ is nontrivial there is n with $0 \le n < k$ such that $\boldsymbol{Q}_1 = \ldots = \boldsymbol{Q}_n = \exists$, and $\boldsymbol{Q}_{n+2} = \ldots = \boldsymbol{Q}_k = \forall$ $(k \ge 2)$.*

Proof. 'If': We have to find $\boldsymbol{Q}_1',\ldots,\boldsymbol{Q}_k'$ such that $B(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k) = \boldsymbol{Q}_1'\ldots\boldsymbol{Q}_k'$. If M is such that $B(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)$ is trivial, clearly this is possible. Otherwise, we take $\boldsymbol{Q}_i' = \boldsymbol{Q}_i$, where $\boldsymbol{Q}_1 = \ldots = \boldsymbol{Q}_n = \exists$, *and* $\boldsymbol{Q}_{n+2} = \ldots = \boldsymbol{Q}_k = \forall$, on M.

[7]Westerståhl 1986 discusses order indpendence of linear prefixes. For MON↑ $\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k$, assuming ISOM and finite models, the prefix $Q_1x_1\ldots Q_kx_k$ is order independent essentially only when $Q_1 = \ldots = Q_k = \forall$ or $Q_1 = \ldots = Q_k = \exists$.

From Lemma 5.5, we know that $B(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)R \Rightarrow \boldsymbol{Q}_1\ldots\boldsymbol{Q}_kR$. For the other direction, suppose that $\boldsymbol{Q}_1\ldots\boldsymbol{Q}_kR$, i.e., that $\exists^{(n)}\cdot\boldsymbol{Q}_{n+1}\cdot\forall^{(k-n-1)}R$. Then

$$\begin{aligned}&\{(a_1,\ldots,a_n) : \boldsymbol{Q}_{n+1}\cdot\forall^{(k-n-1)}R_{a_1\ldots a_n}\}\\ =\ &\{(a_1,\ldots,a_n) : \boldsymbol{Q}_{n+1}\{b : R_{a_1\ldots a_nb} = M^{k-n-1}\}\} \neq \emptyset.\end{aligned}$$

Take $(a_1,\ldots,a_n)$ in this set. It follows that

$$\{a_1\} \times \cdots \times \{a_n\} \times \{b : R_{a_1\ldots a_nb} = M^{k-n-1}\} \times M \times \cdots \times M \subseteq R.$$

Thus, $B(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)R$.

'Only if': Suppose $B(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k) = \boldsymbol{Q}_1'\ldots\boldsymbol{Q}_k'$, and that $B(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)$ is non-trivial on M. By Lemma 5.3, each $\boldsymbol{Q}_i$ is nontrivial on M (mention of M will be omitted in what follows). By Lemma 5.4 and the assumption, $\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k$ and $\boldsymbol{Q}_1'\ldots\boldsymbol{Q}_k'$ are equal on products. Hence, by the Product Theorem, $\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k = \boldsymbol{Q}_1'\ldots\boldsymbol{Q}_k'$. That is, $B(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k) = \boldsymbol{Q}_1\ldots\boldsymbol{Q}_k$. Now we are in a position to show that the quantifiers $\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k$ have the required form, by induction on k.

INDUCTION BASE, $k = 2$: Suppose $B(\boldsymbol{Q}_1,\boldsymbol{Q}_2) = \boldsymbol{Q}_1\boldsymbol{Q}_2$ on $M = \{a_1,\ldots,a_m\}$. Let n_i be the smallest number in $\boldsymbol{Q}_i$, $i = 1,2$. $n_1,n_2 > 0$, by nontriviality. If $n_1 = 1$, then $\boldsymbol{Q}_1 = \exists$, and we are done. Assume $n_1 > 1$. We must show that $n_2 = m$ (so $\boldsymbol{Q}_2 = \forall$). Suppose instead that $n_2 < m$. Let

$$R = (\{a_1,\ldots,a_{n_1-1}\}\times\{a_1,\ldots,a_{n_2}\}) \cup (\{a_{n_1}\}\times\{a_2,\ldots,a_{n_2},a_m\}).$$

It follows that if $X\times Y \subseteq R$, then either $|X| < n_1$ or $|Y| < n_2$. Thus, $\neg B(\boldsymbol{Q}_1,\boldsymbol{Q}_2)R$. On the other hand, $|\{a\colon |R_a| \geq n_2\}| = n_1$, so $\boldsymbol{Q}_1\boldsymbol{Q}_2R$. This contradicts our assumption.

INDUCTION STEP: Suppose the result is true for k, and $B(\boldsymbol{Q}_0,\ldots,\boldsymbol{Q}_k) = \boldsymbol{Q}_0\ldots\boldsymbol{Q}_k$. We use the following

5.9. Lemma. *If $B(\boldsymbol{Q}_0,\ldots,\boldsymbol{Q}_k) = \boldsymbol{Q}_0\ldots\boldsymbol{Q}_k$ and $B(\boldsymbol{Q}_0,\ldots,\boldsymbol{Q}_k)$ is nontrivial on M, then $B(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k) = \boldsymbol{Q}_1\ldots\boldsymbol{Q}_k$ and $B(\boldsymbol{Q}_0,\ldots,\boldsymbol{Q}_{k-1}) = \boldsymbol{Q}_0\ldots\boldsymbol{Q}_{k-1}$.*

Proof. Fix $A \subseteq M$ such that $\boldsymbol{Q}_0A$ (nontriviality and Lemma 5.3) and take any $R \subseteq M^k$. We have

$$\begin{aligned}&B(\boldsymbol{Q}_0,\ldots,\boldsymbol{Q}_k)A\times R\\ \Leftrightarrow\ &\exists X_0,\ldots,X_k\,[\boldsymbol{Q}_0X_0\ \&\ \ldots\ \&\ \boldsymbol{Q}_kX_k\ \&\ X_0\times\cdots\times X_k \subseteq A\times R]\\ \Leftrightarrow\ &\exists X_0[\boldsymbol{Q}_0X_0\ \&\ X_0 \subseteq A]\ \&\\ &\exists X_1,\ldots,X_k\,[\boldsymbol{Q}_1X_1\ \&\ \ldots\ \&\ \boldsymbol{Q}_kX_k\ \&\ X_1\times\cdots\times X_k \subseteq R]\\ \Leftrightarrow\ &\boldsymbol{Q}_0A\ \&\ B(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)R\\ \Leftrightarrow\ &B(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)R.\end{aligned}$$

But also,

$$\boldsymbol{Q}_0\ldots\boldsymbol{Q}_k A\times R$$
$$\Leftrightarrow \quad \boldsymbol{Q}_0 A \ \& \ \boldsymbol{Q}_1\ldots\boldsymbol{Q}_k R \qquad \text{(product decomposition)}$$
$$\Leftrightarrow \quad \boldsymbol{Q}_1\ldots\boldsymbol{Q}_k R.$$

This proves the first part of the lemma. The second part is proved by fixing B such that $\boldsymbol{Q}_k B$ and considering $R\times B$. —|

To finish the proof of Theorem 5.9, the induction hypothesis and the lemma show that both $B(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)$ and $B(\boldsymbol{Q}_0,\ldots,\boldsymbol{Q}_{k-1})$ have the desired 'form' on M. But then it is readily verified that the same holds for $B(\boldsymbol{Q}_0,\ldots,\boldsymbol{Q}_k)$. —|

6. Cumulation

So-called *cumulative readings* (Scha 1981) of quantifiers are natural for sentences like

(1) Sixty teachers taught seventy courses at the summer school
(2) Five girls told ten stories to three boys.

(2) has a reading involving five girls, ten stories, and three boys, but not saying exactly how many stories the first girl told the second boy etc., only that each girl told some story to some boy, that each story was told by some girl to some boy, etc. This leads to the following

6.1. Definition.

$$(Q_1,\ldots,Q_k)^c x_1\ldots x_k R x_1\ldots x_k \quad \leftrightarrow$$
$$Q_1 x_1 \exists x_2 \ldots \exists x_k R x_1 \ldots x_k \wedge \ldots \wedge Q_k x_k \exists x_1 \ldots \exists x_{k\text{-}1} R x_1 \ldots x\ .$$

Or, equivalently,

(3) $$(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)^c R \ \Leftrightarrow\ \bigwedge\nolimits_{1\le i\le k} \boldsymbol{Q}_i\exists\ldots\exists R^{(i,1,\ldots,i-1,i+1,\ldots,k)}.$$

Thus, cumulatives are Boolean combinations of iterations, but we shall see that they are very seldom iterations themselves. We only consider cumulations of positive (but not necessarily monotone) quantifiers, and start with the following observations.

6.2. Lemma. *Let $\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k$ be positive. Then $(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)^c$ and $\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k$ are equal on products.*

Proof.

$$
\begin{aligned}
&(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)^{\mathrm{c}}A_1\times\cdots\times A_k \\
\Leftrightarrow\quad & \textstyle\bigwedge_{1\le i\le k} \boldsymbol{Q}_i\,\exists\ldots\exists(A\times A_1\times\cdots\times A_{i-1}\times A_{i+1}\times\cdots\times A_k) \quad \text{(definition)}\\
\Leftrightarrow\quad & \textstyle\bigwedge_{1\le i\le k} (\boldsymbol{Q}_iA_i \;\;\&\;\; \bigwedge_{j\neq i}(A_j\neq\varnothing)) \quad \text{(product decomposition)}\\
\Leftrightarrow\quad & \boldsymbol{Q}_1A_1 \;\&\; \ldots \;\&\; \boldsymbol{Q}_kA_k \quad \text{(positivity)}\\
\Leftrightarrow\quad & \boldsymbol{Q}_1\ldots\boldsymbol{Q}_kA_1\times\cdots\times A_k \quad \text{(product decomposition).}
\end{aligned}
$$

—|

6.3. Lemma. *Suppose $\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k$ are positive on M. $(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)^{\mathrm{c}}$ is nontrivial on M iff each $\boldsymbol{Q}_i$ is nontrivial on M.*

Proof. If $\boldsymbol{Q}_j$ is trivial on M it must be empty on M (since it is positive), but then the corresponding conjunct in (3) is always false, so $(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)^{\mathrm{c}}$ is trivial on M. If each $\boldsymbol{Q}_i$ is nontrivial on M there are $A_i\neq\varnothing$ such that $\boldsymbol{Q}_iA_i$, $1\le i\le k$, and thus, just as in the proof of the previous lemma, $(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)^{\mathrm{c}}A_1\times\cdots\times A_k$. We must show that $\neg\forall R\subseteq M(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)^{\mathrm{c}}R$. Otherwise, $(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)^{\mathrm{c}}\varnothing$. But then the first conjunct in (3) is $\boldsymbol{Q}_1\cdot\exists\ldots\exists\varnothing$, and hence $\boldsymbol{Q}_1\varnothing$, contradicting the positivity of $\boldsymbol{Q}_1$. —|

Now we can characterize the (positive) cumulations which are iterations.

6.4. Theorem. *$(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)^{\mathrm{c}}$ is a k-ary iteration iff, on each M where $(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)^{\mathrm{c}}$ is nontrivial, $\boldsymbol{Q}_2=\ldots=\boldsymbol{Q}_k=\exists$.*

Proof. 'If': As usual, we can obviously find the required iteration on universes where $(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)^{\mathrm{c}}$ is trivial. On other universes, one easily verifies, using the positivity of $\boldsymbol{Q}_1$, that $(\boldsymbol{Q}_1,\exists,\ldots,\exists)^{\mathrm{c}}=\boldsymbol{Q}_1\cdot\exists\ldots\exists$.

'Only if': Suppose $(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)^{\mathrm{c}}=\boldsymbol{Q}_1'\ldots\boldsymbol{Q}_k'$, and that $(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)^{\mathrm{c}}$ is nontrivial on M. By Lemma 6.2, $\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k$ and $\boldsymbol{Q}_1'\ldots\boldsymbol{Q}_k'$ are equal on products, and hence equal, by the Product Theorem. Thus, $(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)^{\mathrm{c}}=\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k$ on $M=\{a_1,\ldots,a\}$. We shall prove the

CLAIM: If $0<n_i<m$ then $n_i\in\boldsymbol{Q}_i \Leftrightarrow n+1\in\boldsymbol{Q}_i$, for $i=2,\ldots,k$.

Since $0\notin\boldsymbol{Q}_i$ but some $p\in\boldsymbol{Q}_i$, it follows from the Claim that $\boldsymbol{Q}_i=\exists$, for $i=2,\ldots,k$.

To prove the Claim, fix such an i. For each $j\neq i$, $1\le j\le k$, choose $n_j\in\boldsymbol{Q}_j$. Thus, each $n_j>0$. Choose any n_i with $0<n_i<m$. We define the k-ary relation R by the following stipulations:

$$Rx_1\ldots x_{i-1}a_1x_{i+1}\ldots x_k \;\;,\;\ldots\;,\;\; Rx_1\ldots x_{i-1}a_nx_{i+1}\ldots x_k \;\;,$$

for all $x_1 \in \{a_1,\ldots,a_{n_1-1}\}$ and $x_j \in \{a_1,\ldots,a_{n_j}\}$, $2 \le j \le k, j \ne i$,

and $Ra_{n_1}x_2\ldots x_{i-1}a_2x_{i+1}\ldots x_k$, … , $Ra_{n_1}x_2\ldots x_{i-1}a_{n_i+1}x_{i+1}\ldots x_k$,
for all $x_j \in \{a_1,\ldots,a_{n_j}\}$, $2 \le j \le k, j \ne i$,

and no other k-tuples are in R.

Then R has the following properties.

(i) For $j \ne i$, $1 \le j \le k$:
$\{y_j : \exists x_1\ldots\exists x_{j-1}\exists x_{j+1}\ldots\exists x_k Rx_1\ldots x_{j-1}y_j\, x_{j+1}\ldots x_k\} = \{a_1,\ldots,a_{n_j}\}$, and hence, $\boldsymbol{Q}_j\cdot\exists\ldots\exists R^{(j,1,\ldots,j-1,j+1,\ldots,k)}$.

(ii) $\{y_i : \exists x_1\ldots\exists x_{i-1}\exists x_{i+1}\ldots\exists x_k Rx_1\ldots x_{i-1}y_i\, x_{i+1}\ldots x_k\} = \{a_1,\ldots,a_{n_i+1}\}$.

(iii) For $i < r < k$:
If $x_j \in \{a_1,\ldots,a_{n_j}\}$ for $j \ne i$, $1 \le j < r$, and $x_i \in \{a_1,\ldots,a_{n_i+1}\}$, then $\{y_r : \boldsymbol{Q}_{r+1}\ldots\boldsymbol{Q}_k R_{x_1\ldots x_{r-1}}y_r\} = \{a_1,\ldots,a_{n_r}\}$. Otherwise, $\{y_r : \boldsymbol{Q}_{r+1}\ldots\boldsymbol{Q}_k R_{x_1\ldots x_{r-1}}y_r\} = \varnothing$.

(i) and (ii) are immediate from the definition of R. (iii) follows by a (downward) inductive argument on r (omitted here), using the fact that $0 \notin \boldsymbol{Q}_r$ but $n_r \in \boldsymbol{Q}_r$, for $i < r \le k$. (iii) is not true for $r = i$. However, we still have

(iv) If $x_j \in \{a_1,\ldots,a_{n_j}\}$ for $1 \le j < i$, then $|\{y_i : \boldsymbol{Q}_{i+1}\ldots\boldsymbol{Q}_k R_{x_1\ldots x_{i-1}}y_i\}| = n_i$. Otherwise, $|\{y_i : \boldsymbol{Q}_{i+1}\ldots\boldsymbol{Q}_k R_{x_1\ldots x_{i-1}}y_i\}| = 0$.

Indeed, it follows from (iii) and the defining conditions of R that $\{y_i : \boldsymbol{Q}_{i+1}\ldots\boldsymbol{Q}_k R_{x_1\ldots x_{i-1}}y_i\}$ is either $\varnothing$ or else $\{a_1,\ldots,a_{n_i}\}$ or $\{a_2,\ldots,a_{n_i+1}\}$.

Now, suppose first that $n_i \in \boldsymbol{Q}_i$. Then, by (iii) and (iv), we can ‘continue downward’, from $r = i$ to $r = 1$, eventually obtaining

$$\{y_1 : \boldsymbol{Q}_2\ldots\boldsymbol{Q}_k R_{y_1}\} = \{a_1,\ldots,a_{n_1}\},$$

and hence $\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k R$. But then, $(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)^c R$, and so in particular,

$$\boldsymbol{Q}_i\cdot\exists\ldots\exists R^{(i,1,\ldots,i-1,i+1,\ldots,k)}.$$

By (ii), this is equivalent to

$$\boldsymbol{Q}_i\{a_1,\ldots,a_{n_i+1}\},$$

and thus $n+1 \in \boldsymbol{Q}_i$.

Now suppose instead $n_i \notin \boldsymbol{Q}_i$. It follows that from step i–1 and downward in the above induction we always get $\varnothing$, and so $\neg\boldsymbol{Q}_1\ldots\boldsymbol{Q}_k R$. Hence,

$\neg(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)^{\mathrm{c}}R$. But, by (i), all the conjuncts in the definition of $(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)^{\mathrm{c}}R$, except the one involving $\boldsymbol{Q}_i$, are true. It follows that

$$\neg\boldsymbol{Q}_i\cdot\exists\ldots\exists R^{(i,1,\ldots,i-1,i+1,\ldots,k)},$$

which, by (ii), means that $n+1 \notin \boldsymbol{Q}_i$. This proves the Claim, and thereby the theorem. $\dashv$

7. Unary complexes

In this section, we briefly look at a generalization of the question as to when a certain quantifier is an iteration. The linguistic interest of this question stems from the ubiquity of natural language sentences with a transitive verb and quantified subject and object noun phrases. But other means of expression are also 'natural'. For one thing, Boolean operators are clearly available. For another, sentences corresponding to iterations are often ambiguous, and hence all their readings can be used. Restricting attention to 2-ary iterations, this leads to the following definition, from van Benthem 1989 (though he uses 'unary complex' for what I here call a 'right complex').

7.1. Definition. A quantifier $\boldsymbol{Q}$ of type <2> is a *unary complex* if there is a Boolean combination Φ of iterations of the form $\boldsymbol{Q}_1\boldsymbol{Q}_2R$ and inverse iterations of the form $\boldsymbol{Q}_1'\boldsymbol{Q}_2'R^{-1}$, such that for all R, $\boldsymbol{Q}R \Leftrightarrow \Phi$. $\boldsymbol{Q}$ is a *right complex* (*left complex*) if only iterations (inverse iterations) are used.

The next proposition is an example of the added expressive power of unary complexes compared to iterations. Note that, by Corollary 4.9, $(\exists_{\geq n})^{(2)}$ is not an iteration for $n \geq 2$.

7.2. Proposition. $(\exists_{\geq n})^{(2)}$ *is a right complex for all n.*

Proof. (sketch) We must express '$|R| \geq n$', for binary R, as a right complex. Start with the following equivalence, which is clearly valid:

$$|R| \geq n \quad \Leftrightarrow \quad \bigvee_{1\leq k\leq n-1} (\exists_{=k}x\exists yRxy \;\wedge\; |R| \geq n) \quad \vee \quad \exists_{\geq n}x\exists yRxy.$$

Thus, it suffices to express each of the n–1 first disjuncts as right complexes. Given that exactly k elements have R-successors, it is not so hard to describe the circumstances under which $|R| \geq n$. Consider the different ways, say $s_1,\ldots,s_r$, in which n can be written as the sum of k positive integers (independent of order). Each such way corresponds to a minimal distribution of successors (over the k elements which have successors) so that $|R| \geq n$. With each s_i we will correlate a right complex ψ_i. Rather than giving precise details, we explain the idea by means of an example. There are 5 ways in which 10 (n) can be written as the sum of 6 (k) positive integers. We exhibit them below, together with the corresponding right complexes:

s_i	ψ_i
5 + 1 + 1 + 1 + 1 + 1	$\exists x \exists_{\geq 5} y Rxy$
4 + 2 + 1 + 1 + 1 + 1	$\exists_{\geq 2} x \exists_{\geq 2} y Rxy \ \wedge\ \exists x \exists_{\geq 4} y Rxy$
3 + 3 + 1 + 1 + 1 + 1	$\exists_{\geq 2} x \exists_{\geq 3} y Rxy$
3 + 2 + 2 + 1 + 1 + 1	$\exists_{\geq 3} x \exists_{\geq 2} y Rxy \ \wedge\ \exists x \exists_{\geq 3} y Rxy$
2 + 2 + 2 + 2 + 1 + 1	$\exists_{\geq 4} x \exists_{\geq 2} y Rxy$

(note that, since $\exists_{=6} x \exists y Rxy$, conjuncts corresponding to the 1's, such as $\exists_{\geq 5} x \exists y Rxy$ for the first row, are not needed in ψ_i). Hopefully the idea is clear, and one may now verify that, in general,

$$\exists_{=k} x \exists y Rxy \ \wedge\ |R| \geq n \quad \Leftrightarrow \quad \exists_{=k} x \exists y Rxy \ \wedge\ (\psi_1 \vee \ldots \vee \psi_r).$$

—|

The properties of *orientation* from van Benthem 1989 provide convenient ways of showing that certain quantifiers are *not* unary (right, left) complexes. For example, in this way one easily sees that cumulations, although unary complexes, are usually not right or left complexes, and that branchings are usually not unary complexes. However, the method does not work for resumptions, since all resumptions have these orientation properties. Our final application of 'prefix techniques' in this paper shows that the resumption of $\boldsymbol{Q}^{\mathrm{R}}$ is not a unary complex. Recall that, on finite universes, $\boldsymbol{Q}^{\mathrm{R}}$ means 'more than half of the elements of the universe'. It is clear that the next result extends to any proportion quantifier 'more than m/n:ths of the elements of the universe'.

7.3. Theorem. *$(\boldsymbol{Q}^{\mathrm{R}})^{(2)}$ is not a unary complex.*

Proof. We shall prove that not even $(\boldsymbol{Q}^{\mathrm{R}})^{(2)} A \times B$ can be expressed by a (fixed) unary complex. Suppose to the contrary that there is a unary complex Φ, i.e., a Boolean combination of iterations of the forms $\boldsymbol{Q}_1 \boldsymbol{Q}_2 A \times B$ and $\boldsymbol{Q}_3 \boldsymbol{Q}_4 B \times A$ such that for all M and all $A, B \subseteq M$,

$$|A \times B| > |M|^2/2 \quad \Leftrightarrow \quad \Phi.$$

We now perform the following operations on Φ. First, by redefining the quantifiers in Φ if necessary, make sure that the second quantifier in an iteration (or inverse iteration) is always positive. Then replace, according to the Product Decomposition Lemma, each $\boldsymbol{Q}_1 \boldsymbol{Q}_2 A \times B$ in Φ by $(\boldsymbol{Q}_1 A\ \wedge$

$\neg Q_2 B) \vee (Q_1\varnothing \wedge \neg Q_2 B)$, and similarly for the inverse iterations. Next, rewrite the result in disjunctive normal form. Finally, in each disjunct, 'pull together' the conjuncts which involve A by defining suitable new quantifiers (for example, if the conditions on A in one conjunct are $Q'A$, $Q''A$, and $\neg Q'''A$, replace $Q'A \wedge Q''A \wedge \neg Q'''A$ by QA , where $Q = Q' \wedge Q'' \wedge \neg Q'''$), and similarly for B and $\varnothing$. The result of all this is that there are type <1> quantifiers Q_i , Q_i', and Q_i'', $1 \le i \le p$, which satisfy ISOM and are such that

$$|A\times B| > |M|^2/2 \quad \Leftrightarrow \quad \bigvee_{1\le i\le p} (Q_i A \wedge Q_i'B \wedge Q_i''\varnothing) \qquad \text{(on } M\text{)}.$$

In other words, for all $m > 0$ there are $X,Y,Z_i \subseteq \{0,...,m\}$, $1 \le i \le p$, such that for all $k,n \le m$,

$$kn > m^2/2 \quad \Leftrightarrow \quad \bigvee_{1\le i\le p} (k \in X_i \wedge n \in Y_i \wedge 0 \in Z_i).$$

Simplifying a little, it follows that

(*) $\exists p\ \forall m > 0\ \exists p' \le p\ \exists X,Y \subseteq \{0,\ldots,m\}$ for $1 \le i \le p'$ s.t. $\forall k,n \le m$,

$$kn > m^2/2 \quad \Leftrightarrow \quad \bigvee_{1\le i\le p'} ((k,n) \in X_i \times Y_i).$$

It is intuitively plausible that (*) cannot possibly be true, since p is fixed but m arbitrary. Nevertheless, here is a proof.

Suppose m is even and large enough (cf. below). For $k,n \le m$, call (k,n) *minimal* if $kn > m^2/2$, but $(k-1)n$, $k(n-1) \le m^2/2$. We shall count the number of minimal pairs.

(i) If (k,n) is minimal then (n,k) is minimal.
(ii) If (k,n) is minimal then $k,n > m/2$ and $k \ne n$.
(iii) $(m/2 + 1 , m - 1)$ is minimal.

These are all immediate or almost: that (k,k) cannot be minimal follows from the fact that if $k\cdot k > m^2/2$ then $k(k-1) > m^2/2$, for large enough m ($m > 70$ suffices).

Let $k_0 = m/2 + 1$, and then let $k_{i+1} = k_i + 1$ until we reach $k_i = k_{l-1} + 1$ = the largest k such that $k\cdot k \le m^2/2$. Also, let n_i be the smallest n such that $k\cdot n > m^2/2$.

(iv) (k_i ,n_i) is minimal, $n_i > k_i$, for $0 \le i \le l$, and $n_0 > \ldots > n_l$.

Proof: That $n_i > k_i$ is immediate. The rest is by induction. $(k_0 ,n_0) = (m/2 + 1 , m-1)$ is minimal. Suppose (k_i ,n_i) is minimal. Then we have $k_{i+1}(n_i - 1) = (k_i + 1)(n_i - 1) = kn_i + n_i - (k_i + 1) \ge kn_i$ (since $n_i > k_i$) $> m^2/2$. Thus, $n_{i+1} < n$. To see that (k_{i+1},n_{i+1}) is minimal, it suffices to check that $(k_{i+1} -$

$1)n_{i+1} \le m^2/2$. But $(k_{i+1} - 1)n_{i+1} = k_{i+1}n_{i+1} - n_{i+1} < k_{i+1}n_{i+1} - k_{i+1} = k_{i+1}(n_{i+1} - 1) \le m^2/2$, by the definition of n_{i+1}.

(v) If (k,n) is minimal and $k < n$, then $(k,n) = (k_i, n_i)$, for some i.

Proof: It suffices to show that $k_0 \le k \le k_l$. Clearly $k_0 \le k$. Suppose $k > k_l$. Now $(k_l + 1)(k_l + 1) > m^2/2$, and hence $(k_l + 1) > m^2/2$. It follows that $(k - 1)n > m^2/2$, contradicting the minimality of (k,n).

From (i), (ii), (iv) and (v) we get

(vi) There are $2(l + 1)$ minimal pairs.

Now we can return to (*). The point of the preceding exercise is this:

(vii) Distinct minimal pairs belong to distinct $X_i \times Y_i$ in (*).

Proof: Suppose (k,n), (k',n') are minimal, $(k,n) \neq (k',n')$, say, $k < k'$, and $(k,n),(k',n') \in X_i \times Y$. But then $(k,n') \in X_i \times Y$, so $kn' > m^2/2$ by (*), which contradicts the minimality of (k',n').

We have shown that $p \ge p' \ge 2(l + 1)$. But this is impossible, since p is fixed and l increases with m, in fact,

(viii) $l > m/2(\sqrt{2} + 1) - 2$.

Proof: Easy calculation, from $(k_l + 1)^2 > m^2/2$, and $k_l = m/2 + l + 1$.

This concludes the proof of the theorem. —|

I have been a bit fussy about distinguishing local from global results in this paper. The usual notion of definability in logic is global, i.e., uniform over universes, and I have endeavoured to state global forms of all definability results here. We have seen that many of these results have both a global and a local version. The last result above, however, is a good illustration of the point that this is not always so. Indeed, it follows from a result in van Benthem 1989 that, *on any given universe M*, $(\boldsymbol{Q}^{\mathrm{R}})^{(2)}$ can be defined as a right complex. There is just no definition that works for all universes.[8]

8. Issues for further study

What else could be said about iteration? Very briefly, here are a few suggestions.

1. *Characterizing quantifier lifts*. Can iteration be characterized in terms of their properties? That is, are there (interesting) properties such that, say, a

---

[8]In Westerståhl 1992 I conjecture that $(\boldsymbol{Q}^{\mathrm{R}})^{(2)}$ is not even monadically definable, that is, not definable in any logic $L(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_k)$, where the $\boldsymbol{Q}_i$ are monadic.

type <2> quantifier is a 2-ary iteration iff it has these properties?[9] One necessary such property is being determined by its behaviour on products. We have seen in Part II that this is not sufficient, but combining it with other properties might give a sufficient condition.

This is part of a more general issue. Iteration, resumption, branching and cumulation can all be considered as natural *liftings* of monadic quantifiers to polyadic ones (or, more generally, liftings of quantifiers of certain types to quantifiers of 'higher' types). These lifts have characteristic properties, and it would be interesting to know if they can be completely characterized in terms of these properties. A similar question is studied in van der Does 1992, 1993 in the field of collective quantification (i.e., quantification over collections, or sets, of individuals). He displays several lifts from ordinary quantification to quantification over collections, and, among other things, characterizes the lifts in terms of their respective properties, such as (versions of) conservativity, monotonicity, etc. Further, in both cases one can study which linguistic mechanisms trigger such lifts. In general (as Johan van Benthem has pointed out), polyadic quantification of the kinds studied here and collective quantification seem to have much in common.

2. *Generalizing the Prefix Theorem*. It is an immediate corollary of the Prefix Theorem that if $Q_1x_1\ldots Q_kx_k$ and $Q_1'x_1\ldots Q_k'x_k$ are prefixes with Q_i and Q_i' in $\{\forall,\exists\}$ such that

$$(*) \qquad \models Q_1x_1\ldots Q_kx_kRx_1\ldots x_k \quad \leftrightarrow \quad Q_1'x_1\ldots Q_k'x_kRx_1\ldots x_k\,,$$

then $Q_i = Q_i'$ for each i (in view of the Product Theorem, we could replace $Rx_1\ldots x_k$ by $P_1x_1 \wedge \ldots \wedge P_kx_k$ here). This is a weak version of the Linear Prefix Theorem in Keisler and Walkoe 1973. In the original version, $Rx_1\ldots x_k$ on the right hand side in (*) is replaced by an arbitrary quantifier-free formula ϕ (without constant or function symbols). As Keenan notes, the strong version does not hold for arbitrary type <1> quantifiers; even $\forall xPx \leftrightarrow (\neg\exists)x\neg Px$ is a counterexample. But it might still hold under some restrictions. For example, does it hold when all the quantifiers are positive?

3. *Infinite universes*. The proofs of the main theorems in Part II depend heavily on the assumption that universes are finite. Are there versions of these results for infinite models?

---

[9]Keenan 1992 has such a characterization (the Reducibility Characterization Theorem), but the property used is too reminiscent of the definition of iteration to be of real interest here (it has other uses, mainly as a tool for showing that certain quantifiers are not iterations).

4. *General branching*. Barwise's branching of monotone quantifiers has been extended to other generalized quantifiers (cf. Spaan 1992 for references and a discussion of the various options here). Can the results in section 5 be extended accordingly?

5. *Iteration in other types*. The notion of iteration is not confined to quantifiers. Thinking of quantifiers as objects in type theory, one may generalize the idea to objects of (certain) other types. What is the common pattern here? Will characteristic properties of quantifier iteration, such as the Product Theorem, carry over to the general case?

Already for the case studied here, the perspective of type theory and categorial grammar, may be fruitful. Johan van Benthem remarked that our basic iteration scheme (Definition 1.3) yields a type transition

$$(e^k \to t) \to t\ ,\ (e^m \to t) \to t \quad \Rightarrow \quad (e^{k+m} \to t) \to t$$

(where $e^1 = e$, $e^{n+1} = e \cdot e^n$) which is provable in the Lambek Calculus, and that the scheme itself is precisely the lambda term for its most straightforward derivation (cf. van Benthem 1991). Likewise, some properties of iterations, like the preservation of ISOM or CONSERV, can be predicted from a categorial analysis.

6. *Syllogistic inference*. Various kinds of syllogistic inference with non-iterated quantifiers are known from the literature. For example, consider a language which has atomic formulas of the form QAB , where A and B are Boolean combinations of *set variables* $X_1, X_1, \ldots$, and Q is a type <1,1> quantifier symbol selected among $Q_1, Q_2, \ldots$, where $\boldsymbol{Q}_1, \boldsymbol{Q}_2, \ldots$ are given quantifiers. The language also has the usual propositional connectives, and the obvious semantics. For simple choices of $\boldsymbol{Q}_1, \boldsymbol{Q}_2, \ldots$, complete axiomatizations of validity are known; for example, van der Hoek and de Rijke 1991 axiomatize the case when Q_i = *at least i* (in this logic, quantifiers like *all* and *no* are of course definable).

This could be generalized to iterations, say, of type <1,1,2>. One then adds variables $R_1, R_2, \ldots$ for binary relations, and in addition to the Boolean operations, one might have other operations from relational algebra, such as *converse*. New 'atomic' formulas are $QQ'AB,R$, with A,B as before, R an expression in the chosen relational algebra, and Q,Q' type <1,1> quantifier symbols. The expressive power has increased somewhat, but is still weak compared to $L(\boldsymbol{Q}_1, \boldsymbol{Q}_2, \ldots)$. Do the valid sentences still have nice axiomatizations for natural choices of $\boldsymbol{Q}_1, \boldsymbol{Q}_2, \ldots$?

References

Barwise, Jon. 1979. On Branching Quantifiers in English. *Journal of Philosophical Logic* 8, 47–80.

Ben-Shalom, Dorit. 1993. A Tree Characterization of Generalized Quantifier Reducibility. This volume.

van Benthem, Johan. 1989. Polyadic Quantifiers. *Linguistics and Philosophy* 12, 437–464.

van Benthem, Johan. 1991. *Language in Action*. Dordrecht: North-Holland.

van der Does, Jaap. 1992. *Applied Quantifier Logics*. Dissertation, University of Amsterdam.

van der Does, Jaap. 1993, On Complex Plural Noun Phrases. This volume.

van der Hoek, Wiebe, and de Rijke, Maarten. 1991. Generalized Quantifiers and Modal Logic. In Jaap van der Does and Jan van Eijck, eds., *Generalized Quantifier Theory and Application*s, 115–142. Institute of Logic, Language and Information, Amsterdam. Also in the *Journal of Logic, Language and Information* 2, 19–58, 1993.

Keenan, Edward L. 1992. Beyond the Frege Boundary. *Linguistics and Philosophy* 15.

Keenan, Edward L. 1993. Natural Language, Sortal Reducibility and Generalized Quantifiers. *Journal of Symbolic Logic* 58, 314–325.

Keisler, Jerome, and Walkoe, Wilbur Jr. 1973. The Diversity of Quantifier Prefixes. *Journal of Symbolic Logic* 38, 79–85.

May, Robert. 1989. Interpreting Logical Form. *Linguistics and Philosophy* 12, 387–435.

Scha, Remko. 1981. Distributive, Collective and Cumulative Quantification. In Jeroen Groenendijk *et al*., eds., *Formal Methods in the Study of Language (Part I & II)*, 483–512. Amsterdam.

Spaan, Martijn. 1992. Parallel Quantification. To appear in Jaap van der Does and Jan van Eijck, eds., *Generalized Quantifier Theory and Applications* (working title). CSLI Lecture Notes.

Westerståhl, Dag. 1986. On the Order between Quantifiers. In Mats Furberg *et al*., eds., *Logic and Abstraction. Essays dedicated to Per Lindström on his Fiftieth Birthday*, 273–285. Acta Philosophica Gothoburgiensia I, University of Göteborg.

Westerståhl, Dag. 1989. Quantifiers in Formal and Natural Languages. In Dov Gabbay and Frans Guenthner, eds., *Handbook of Philosophical Logic, vol. IV*, 1–131. Dordrecht: Reidel.

Westerståhl, Dag. 1992. Quantifiers in Natural Language: a Survey of Some Recent Work. To appear in Michal Krynicki and Marcin Mostowski, eds., *Quantifiers*. Dordrecht: Kluwer.