Quantifiers

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1 Routes to quantifiers

There are two main routes to a concept of (generalized) quantifier. The first starts from first-order logic, *FO*, and generalizes from the familiar ∀ and ∃ occurring there. The second route begins with real languages, and notes that many so called *noun phrases*, a kind of phrase which occurs abundantly in most languages, can be interpreted in a natural and uniform way using quantifiers.

In this chapter I take the first route. One reason is that it leads most directly to a most general notion of a quantifier, subsuming those one finds in natural languages. Another reason is that *FO* is so familiar, and in any case presented in the chapter [Hodges 1999] in this book. Indeed, I will assume acquaintance with [Hodges 1999] and (with few exceptions) use the notation introduced there. At end of the chapter I will indicate what quantifiers have to do with natural languages.

The actual historical development of the concept of a quantifier is slightly complicated. The expressions *all, some, no, not all* from Aristotelian syllogistics are readily seen as (generalized) quantifiers of type $\langle 1, 1\rangle$: they are *definable* from ∀ and ∃ but not the same as these; all of this will be explained shortly. Frege, who if anyone must be regarded as the inventor of *FO*, actually had in his posession essentially the concept of a generalized quantifier that we shall encounter here (the main difference being that he quantified over a fixed universe of *all* objects, whereas our quantifiers are relativized to arbitrary (sub)universes). However, since he could express all the mathematics he needed with ∀ and ∃, he was content to have only these (in fact only ∀) in his Begriffsschrift. Much later, when first [Mostowski 1957] and then [Lindström 1966] introduced generalized quantifiers into mathematical logic, opening up the study of so called model-theoretic logics, they were apparently unaware of Frege's notion. Later still, linguists noted the relevance of quantifiers to natural languages, for example, [Barwise and Cooper 1981] and [Keenan and Stavi 1986]. They found, of course, that the four Aristotelian quantifiers were prime examples of 'natural language quantifiers', but also that there were many more, not definable from these.

I will not dwell further on history here; the interested reader can find more in [Westerståhl 1989], which also has a much more detailed presentation of the logical and linguistic properties of quantifiers. Another survey, emphasizing the link to natural languages, is [Keenan and Westerståhl 1997].

2 First-order logic revisited

From [Hodges 1999] we first recall that a first-order language has a *signature* σ which is a set of non-logical symbols: relations symbols $P, R, \ldots$ of various arities, function symbols $F, G, \ldots$ of various arities, and individual constants $c, d, \ldots$. A *structure* (or model) for σ, or simply a σ-structure, consists of a universe A and an appropriate interpretation $\cdot^A$ of the symbols in σ, so that if P is an n-ary relation symbol in σ then P^A is an n-ary relation over A; if F is an n-ary function symbol in σ then F^A is an n-ary operation on A; and if c is an individual constant in σ then c^A is an element of A. So we may write

$$A \;=\; (A, P^A, R^A, \ldots, F^A, G^A, \ldots, c^A, d^A, \ldots,)$$

Note that I use $A, B, \ldots$ for structures where [Hodges 1999] uses $I, J, \ldots$ instead, and moreover that I often use, to save notation, the very *same* letters for the universes of those structures.

The signature and its symbols can often be left implicit. For example, if we write

$$\mathcal{N} \;=\; (N, <, +, \cdot, S, 0),$$

where $N = \{0, 1, 2, \ldots\}$, it is understood that this is a structure for a signature with one binary relation symbol denoting the usual order of the natural numbers, two binary function symbols denoting addition and multiplication respectively, one unary function symbol denoting the successor operation, and one individual constant denoting 0. In fact, one often uses '$<$', '$+$', etc. for both the symbols and their denotations in such a case.

I will call a structure *relational* if its signature contains only relation symbols.

We now have the fundamental relation

(1) $$A \models \psi[a_1, \ldots, a_n],$$

meaning that ψ is true in A under a valuation v such that $v(x_i) = a_i$ for $1 \leq i \leq n$, where $\psi = \psi(x_1, \ldots, x_n)$ is a σ-formula with at most $x_1, \ldots, x_n$ free, A is a σ-structure, and $a_1, \ldots, a_n \in A$. When ψ is a *sentence*, i.e., a formula without free variables, $A \models \psi$ is often read 'ψ is true in A', or 'A is a model of ψ'.

We may abbreviate a sequence $\langle a_1, \ldots, a_n \rangle$ as $\mathbf{a}$. Then, with an obvious extension (or, if you will, abuse) of the above notation, we can write the standard explications of the universal and existential quantifiers as follows, where $\varphi = \varphi(x, x_1, \ldots, x_n)$:

(2) $A \models (\forall x)\varphi[x, \mathbf{a}]$ iff for every $a \in A$, $A \models \varphi[a, \mathbf{a}]$

(3) $A \models (\exists x)\varphi[x, \mathbf{a}]$ iff for some $a \in A$, $A \models \varphi[a, \mathbf{a}]$.

3 What do quantifier symbols denote?

(2) and (3) tell us what '$\forall$' and '$\exists$' mean, but they do so in an indirect way: they do not tell us what, if anything, they *denote*. On the other hand, the structure A does tell us what the symbols in σ denote: P denotes P^A, etc. With a medieval term, the σ-symbols are given *categorematically*, whereas $\forall$ and $\exists$ are defined *syncategorematically*. Can we give a categorematic definition of the quantifiers?

This has been a vexed question in the history of logic. Informally, one might try to think of something like *a man* denoting some particular man. What then about *every man* — it would seem to have to denote the set of all men. But matters get worse if we consider *no men*; does this denote the empty set? If so, it has the same denotation as *no dogs* — this seems wrong. Considerations like these may lead one to suppose that there is no coherent and uniform way of assigning denotations to quantified phrases. But in fact there is, and the theory of generalized quantifiers provides the solution.

Consider first the corresponding question for the propositional operators, say, conjunction. Everyone knows that '&' can be taken to denote a binary *truth function*. The corresponding clause in the usual truth definition does not mention this truth function explicitly, however; it is still syncategorematic:

(4) $$A \models (\varphi \,\&\, \psi)[\mathbf{a}] \text{ iff } A \models \varphi[\mathbf{a}] \text{ and } A \models \psi[\mathbf{a}].$$

To reformulate this, we begin by noting that in a structure A, a formula with k free variables denotes a k-ary relation over A: the set of k-tuples of elements of A satisfying the formula. Thus define, for any σ-formula $\varphi = \varphi(x_1, \ldots, x_n)$, any σ-structure A, and any n-tuple $\mathbf{a}$ of elements in A,

$$\varphi^{A,\mathbf{a}} = \begin{cases} T & \text{if } A \models \varphi[\mathbf{a}] \\ F & \text{otherwise.} \end{cases} \tag{5}$$

Then we can rewrite (4) as

$$A \models (\varphi \,\&\, \psi)[\mathbf{a}] \text{ iff } \&\,(\varphi^{A,\mathbf{a}}, \psi^{A,\mathbf{a}}) = T$$

(or even more compactly as $(\varphi \,\&\, \psi)^{A,\mathbf{a}} = \&\,(\varphi^{A,\mathbf{a}}, \psi^{A,\mathbf{a}})$), where the last '&' denotes the truth function given by the usual truth table for conjunction.

To do something similar for $\forall$ and $\exists$ we extend the notation in (5) as follows. Let A be a σ-structure, $\varphi = \varphi(x, x_1, \ldots, x_n)$ a σ-formula with at most the free variables shown, and $\langle a_1, \ldots, a_n \rangle = \mathbf{a}$ an n-tuple of elements of A. Then

$$\varphi^{A,x,\mathbf{a}} = \{a \in A : A \models \varphi[a, \mathbf{a}]\}.$$

In a structure A, a formula with one free variable denotes a set: the set of objects in A satisfying the formula. If φ has additional free variables $x_1, \ldots, x_n$, but these are interpreted as $a_1, \ldots, a_n$, respectively, then, relative to this interpretation, φ still denotes a set, and this set is $\varphi^{A,x,\mathbf{a}}$. Now we may rewrite (2) and (3) as

(6) $A \models (\forall x)\varphi[x, \mathbf{a}]$ iff $\varphi^{A,x,\mathbf{a}} = A$

(7) $A \models (\exists x)\varphi[x, \mathbf{a}]$ iff $\varphi^{A,x,\mathbf{a}} \neq \emptyset$.

Just one small further step is needed. Let, on each universe A, $\exists_A$ be the set of non-empty subsets of A. And let $\forall_A$ be simply $\{A\}$. Then (6) and (7) become

(8) $A \models (\forall x)\varphi[x, \mathbf{a}]$ iff $\varphi^{A,x,\mathbf{a}} \in \forall_A$

(9) $A \models (\exists x)\varphi[x, \mathbf{a}]$ iff $\varphi^{A,x,\mathbf{a}} \in \exists_A$.

That is, we may think of $\forall$ and $\exists$ as denoting, on a universe A, a *set of subsets* of A. But then, we can call *any* such set of subsets a (generalized) quantifier on A.

For example, suppose we want a quantifier that says "there exist at least n objects such that". Introduce a symbol '$\exists_{\geq n}$' and define, for each universe A,

$$(\exists_{\geq n})_A = \{X \subseteq A : |X| \geq n\}$$

($|X|$ is the cardinality of X). Then the clause

(10) $A \models (\exists_{\geq n} x)\varphi[x, \mathbf{a}]$ iff $\varphi^{A,x,\mathbf{a}} \in (\exists_{\geq n})_A$ iff $|\varphi^{A,x,\mathbf{a}}| \geq n$

gives us just what we want.

The pattern is clear, and completely general. That is, a *quantifier Q on A* is a set of subsets of A. We can also think of 'Q' as a new symbol, such that whenever φ is a formula, so is

$$(Qx)\varphi.$$

(Qx) binds free occurrences of x in φ just as usual, and its meaning is given by the clause

$$A \models (Qx)\varphi[x, \mathbf{a}] \text{ iff } \varphi^{A,x,\mathbf{a}} \in Q_A.$$

Here are some more examples:

(11) $\exists_{=n} = \{X \subseteq A : |X| = n\}$ ("there are exactly n objects such that")

(12) $Q_0 = \{X \subseteq A : X \text{ is infinite}\}$ ("there are infinitely many objects such that"; the name 'Q_0' is standard and is due to the fact that the quantifier means 'at least $\aleph_0$')

(13) $Q_C = \{X \subseteq A : |X| = |A|\}$ (the 'Chang quantifier'; it means $\forall$ on finite sets but *not* on infinite sets)

(14) $Q_R = \{X \subseteq A : |X| > |A - X|\}$ (the 'Rescher quantifier'; on finite sets it means 'for more than half the elements of the universe').

To see the use of such quantifiers, here is a prime example of how we can *express new things* with them. Consider again the structure $\mathcal{N}$ from section 2. It is a fact about this structure that every element has a finite number of predecessors. There is no way to express this in FO — we will see a proof of this later. But using Q_0, the sentence

$$(\forall x) \sim (Q_0 y)(y < x)$$

says exactly this.

4 Monadic quantifiers

Now that we have considered quantifiers which are sets of subsets on a universe A, it is natural to go further and consider *relations* between subsets of A. It is here that we find, to begin, the four Aristotelian quantifiers:

$all_A XY$ iff $X \subseteq Y$ (i.e. if all X are Y, where $X, Y \subseteq A$)

$some_A XY$ iff $X \cap Y \neq \emptyset$

$no_A XY$ iff $X \cap Y = \emptyset$

$not\ all_A XY$ iff $X \not\subseteq Y$.

But there are many more binary relations between subsets of A, for example:

(15) $I_A XY$ iff $|X| = |Y|$ (the Härtig quantifier)

(16) $more_A XY$ iff $|X| > |Y|$

(17) $most_A XY$ iff $|X \cap Y| > |X - Y|$ (on finite universes this means 'more than half of the X are Y')

(18) *at least* $m/n_A XY$ iff $|X \cap Y| \geq m/n \cdot |X|$ ($0 < m < n$; the properly proportional quantifiers — they only make sense if X is finite).

These are just examples: if A has n elements, there are 2^n subsets of A, and 2^{4^n} binary relations between subsets of A. So over a universe with just 2 elements, there are $2^{16} = 65536$ such relations!

We shall say that the quantifiers from section 3 are of *type* $\langle 1 \rangle$, and those considered so far in this section of type $\langle 1, 1 \rangle$. We can go on to consider quantifiers of type $\langle 1, 1, 1 \rangle$, i.e., ternary relations between subsets of the universe, for example,

(19) *more than*$_A XYZ$ iff $|X \cap Z| > |Y \cap Z|$ (more X's than Y's are Z).

In general, a *monadic* quantifier of type $\langle 1, \ldots, 1 \rangle$ on A (with k 1's) is a k-ary relation between subsets of A, for some $k \geq 1$. This terminology indicates that there are also *polyadic* quantifiers, for example of type $\langle 2, 1, 3 \rangle$, but we will leave those until section 11.

Finally, we note that the meaning of a quantifier like *some* or *most* is not dependent on a particular universe; rather it associates with *each* universe a corresponding quantifier *on* that universe. So we have the following general definition:

> A (monadic) quantifier of type $\langle 1, \ldots, 1 \rangle$ (with k 1's) is a function Q which associates with each universe A a quantifier Q_A of type $\langle 1, \ldots, 1 \rangle$ on A, in other words, a k-ary relation between subsets of A.

Such a quantifier Q can also be considered as a variable-binding operator, but now it operates on k formulas and binds one variable in each. That is;

(Q-syn) if $\varphi_1, \ldots, \varphi_k$ are formulas, then $(Qx)(\varphi_1, \ldots, \varphi_k)$ is a formula (where all free occurrences of x in each φ_i are bound by (Qx)),

whose meaning is given by the clause

(Q-sem) $A \models (Qx)(\varphi_1, \ldots, \varphi_k)[\mathbf{a}]$ iff $(\varphi_1^{A,x,\mathbf{a}}, \ldots, \varphi_k^{A,x,\mathbf{a}}) \in Q_A$.

5 Quantifiers and quantities

The quantifiers considered so far have an important feature: they deal only with *quantities*. By contrast, here is an example of a type $\langle 1 \rangle$ quantifier that does *not* deal with quantities. Let John be an individual and define

$$(Q_{John})_A X \text{ iff John} \in X.$$

That is, if John $\in A$ then $(Q_{John})_A$ consists of all those subsets of A containing John; otherwise $(Q_{John})_A$ is empty. This is not an unreasonable object (when John $\in A$). In mathematics, it is called the *principal filter* (over A) generated by John. In linguistics, it has been used to interpret the proper name *John*. But clearly, it says nothing about quantities.

To explain this we need the concept of *isomorphism* between structures. Intuitively, isomorphic structures 'have the same structure' and can for many purposes be identified. Let A and B be structures for the same signature σ, which we, for simplicity, can take to be relational. An *isomorphism* between A and B is a bijection f from the universe A to the universe B (a one-one mapping from one onto the other) such that if P is an n-ary relation symbol in σ and $a_1, \ldots, a_n \in A$, then

$$\langle a_1, \ldots, a_n \rangle \in P^A \iff \langle f(a_1), \ldots, f(a_n) \rangle \in P^B.$$

We write this $f : A \cong B$, and we write $A \cong B$ to say that A and B are isomorphic, i.e., that there is an isomorphism between A and B.

First-order logic cannot distinguish between isomorphic structures:

(*Isomorphism closure*) If $A \cong B$ then every FO sentence which is true in A is true in B, and vice versa.

(The converse of this is far from true in general, though it does hold for finite structures.) In fact, isomorphism closure is usually a requirement on any logic, as we shall see.

For the moment, however, I want to bring out the connection between isomorphism and quantity. First, note that if $A \cong B$ then $|A| = |B|$, since the latter means by definition that there is a bijection between A and B. But in the special case of *monadic* structures, i.e., structures with only unary relations, we can say more. Let us consider a signature with two unary relation symbols. A structure $A = (A, X, Y)$ for this signature partitions the universe into 4 parts:

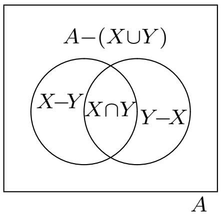


Figure 1.

Now if $f : (A, X, Y) \cong (A', X', Y')$ then the corresponding parts of the two structures must have the same cardinality. For the restriction of the bijection f to, say, $X-Y$ becomes a bijection between $X-Y$ and $X'-Y'$, and similarly for the other parts. But the converse holds too: if the corresponding parts have the same cardinality, then there are four bijections, whose union is an isomorphism between the two structures. That is, we have the following

5.1 Fact. *$(A, X, Y) \cong (A', X', Y')$ iff $|X - Y| = |X' - Y'|$, $|X \cap Y| = |X' \cap Y'|$, $|Y - X| = |Y' - X'|$, and $|A - (X \cup Y)| = |A' - (X' \cup Y')|$.*

This generalizes to all monadic structures: if there are k unary relation symbols in the signature, the universe is partitioned into 2^k parts, and the number of elements in these is, up to isomorphism, all there is to say about the structure.

Now consider the following property of a type $\langle 1, 1\rangle$ quantifier Q:

(ISOM) If $(A, X, Y) \cong (A', X', Y')$ then $[Q_A XY \Leftrightarrow Q_{A'} X'Y']$.

This is what I mean by saying that Q deals only with quantities. If Q satisfies ISOM then, by the above Fact, only the number of elements in $X - Y$, $X \cap Y$, $Y - X$, and $A - (X \cup Y)$ determines whether $Q_A XY$ holds or not. Now look at our examples of type $\langle 1, 1\rangle$ quantifiers from section 4: each one is given by a condition on one or more of these quantities; hence they all satisfy ISOM. For example,

$$all_A XY \iff |X - Y| = 0$$

$$some_A XY \iff |X \cap Y| > 0$$

$$most_A XY \iff |X \cap Y| > |X - Y|$$

$$more_A XY \iff |X - Y| + |X \cap Y| > |Y - X| + |X \cap Y|$$

etc.

ISOM is expressed similarly for other monadic types. In the type $\langle 1 \rangle$ case, it says that whether $Q_A X$ holds or not is determined by the two quantities $|X|$ and $|A - X|$. Thus, all the type $\langle 1 \rangle$ quantifiers from section 3 satisfy ISOM, but the quantifier Q_{John} does not: we may have $X, X' \subseteq A$, $|X| = |X'|$ and $|A - X| = |A - X'|$, but John $\in X - X'$.

It should come as no surprise that there is a tight connection between ISOM and isomorphism closure. To state it, we first need to sharpen our idea of what a *logic* is.

6 Logics with generalized quantifiers

In [Hodges 1999], first-order logic FO is characterized as a collection of (artificial) languages: for each signature σ we have the set of σ-formulas, defined inductively, starting with the atomic formulas, and then one clause for each logical constant. Then, the relation $\models$ between a σ-structure, a σ-formula, and a valuation (of the variables in the universe of the structure) is defined with a corresponding induction, with (2) and (3) (or (8) and (9)) as the inductive clauses corresponding to $\forall$ and $\exists$.

Now let Q be any (for the time being monadic) quantifier. The logic $FO(Q)$ is given, syntactically, by adding (Q-syn) (cf. section 4) as a defining clause of the σ-formulas, and, semantically, (Q-sem) as a defining clause for $\models$. Thus, $FO(Q)$ has all the expressive machinery of first-order-logic, *plus* the quantifier Q.

Similarly, we can define $FO(Q_1, \ldots, Q_n)$, or even $FO(\mathbf{Q})$ where $\mathbf{Q}$ is any set of quantifiers. By a *logic* I will mean a logic of this form (there are more general notions of a logic but they will not concern us here).

For example, $FO(Q_0)$ is a logic, with atomic formulas, negations, conjunctions, existential and universal quantifications as usual, *and* formulas of the form

$$(Q_0 x)\varphi,$$

whose meaning is given by

$$A \models (Q_0 x)\varphi[x, \mathbf{a}] \iff \varphi^{A,x,\mathbf{a}} \text{ is infinite.}$$

$FO(Q_0, \exists_{=17})$ is another logic, where we in addition have formulas of the form

$$(\exists_{=17} y)\varphi,$$

where

$$A \models (\exists_{=17} y)\varphi[y, \mathbf{a}] \iff |\varphi^{A,y,\mathbf{a}}| = 17.$$

The notion of isomorphism closure makes sense for any logic, and it is now easy to establish

6.1 Fact. *If each quantifier in* $\mathbf{Q}$ *satsifies ISOM, then ismorphism closure holds for* $FO(\mathbf{Q})$.

(To show this, one proves by induction over formulas something a little more general, namely, that if $f : A \cong B$ and $a_1, \ldots, a_n \in A$ then $A \models \varphi[a_1, \ldots, a_n] \Leftrightarrow B \models \varphi[f(a_1), \ldots, f(a_n)]$ for all φ in $FO(\mathbf{Q})$ of the relevant signature. Fact 6.1 is the special case of this when φ is a sentence.)

One reason that ISOM is important is thus that we want Fact 6.1 to hold, or, put differently, we want quantifiers to be *logical constants*. There has been some discussion as to just what logicality means, but it is generally agreed that isomorphism closure is at least a necessary condition: logic should be indifferent to which universe of objects we are talking about. It is 'topic-neutral', it cares only about structure. In the case of monadic quantifiers there is a further reason, as we have seen: these particular logical constants care only about quantities of things, not the things in themselves. Hence the adequacy of the term *quantifier*.

Are logics with generalized quantifiers *first-order* or not? There is a sense in which they are: they quantify only over individuals of the universe, not, as in second-order logic, over sets of such individuals. Thus, the notion of a signature, and the notion of a structure, are the same for these logics as for FO. However, the term *first-order logic* has become synonymous with FO, and in this sense many of the logics we have introduced here are not first-order, since their expressive power exceeds that of FO.

7 Expressive power

Consider again the logic $FO(Q_0)$. Clearly this logic, and any logic of the form $FO(\mathbf{Q})$, *extends* FO: everything that can be said in FO can also be said in them. Moreover, it is also clear that $FO(Q_0)$ is *more expressive* than FO; for example, as mentioned in section 3 we can say in $FO(Q_0)$ that every element of $\mathcal{N}$ has a finite number of predecessors, but we cannot say the same thing in FO. Or, to take a simpler example, we can say in $FO(Q_0)$, but not in FO, that the universe is finite:

$$\sim (Q_0 x)(x = x)$$

[One proof of the last claim goes as follows. Suppose there were an FO-sentence θ equivalent to $\sim (Q_0 x)(x = x)$. In FO we can write down, for every natural number n, a sentence φ_n saying that the universe has at least n elements. Consider the theory (set of sentences)

$$T = \{\theta\} \cup \{\varphi_n : n = 1, 2, \ldots\}.$$

Now T has the property that every one of its finite subsets has a model (why?). By the *Compactness theorem* (Corollary 3 in [Hodges 1999]), which holds for FO, it then follows that T has a model. But that is impossible: the universe of that model would be finite, yet have at least n elements for every n. Hence, $\sim (Q_0 x)(x = x)$ is not equivalent to any FO-sentence. Also, it follows that the Compactness theorem does not hold for $FO(Q_0)$.]

We take these intuitions as the way to compare the expressive power of logics. By definition, a logic L' *extends* a logic L, in symbols, $L \leq L'$, if each L-sentence is equivalent to — has the same models as — some L'-sentence (of the same signature). Thus, every logic of the kind we consider here extends FO. Moreover, L' *properly* extends L, $L < L'$, if $L \leq L'$ and $L' \not\leq L$. The latter condition means that there is some L'-sentence which is not equivalent to any L'-sentence. For example,

(20) $$FO < FO(Q_0),$$

as we just saw. Finally, we say that L and L' are *equivalent*, $L \equiv L'$, if $L \leq L'$ and $L' \leq L$.

Note that equivalence between logics means same expressive power; it does not mean identity. Consider FO and $FO(\exists_{=17})$. These logics are equivalent: in FO we can say, for example, that a set has exactly 18 elements:

$$(\exists x_1)\ldots(\exists x_{18})(\bigwedge_{1\leq i\neq j\leq 18} (P(x_i) \,\&\, (x_i \neq x_j)) \,\&\, (\forall y)(P(y) \supset \bigvee_{1\leq i\leq 18} (y = x_i)))$$

As we see, it takes 19 variables to say this. But in $FO(\exists_{=17})$ we manage to say the same thing with just 2 variables:

$$(\exists y)(P(y) \,\&\, (\exists_{=17} x)(P(x) \,\&\, (x \neq y))).$$

Indeed, everything that can be said in $FO(\exists_{=17})$ can also be said in FO, only it sometimes takes more variables. The number of variables used is important for certain applications of logic, but not for expressive power as defined here.

8 Definability

Showing that $L \leq L'$ may seem like a substantial task: we must find for each one of the infinitely many L-sentences an equivalent L'-sentence. But when L is of the form $FO(\mathbf{Q})$, the task is usually much simpler: it suffices to show that each quantifier in $\mathbf{Q}$ is *definable* in L'. For example, once we see that the quantifier $\exists_{=17}$ is definable in FO, it is rather clear that any $FO(\exists_{=17})$-sentence can be rewritten as an FO-sentence. And that $\exists_{=17}$ is definable in FO just means that the single sentence

$$(\exists_{=17}x)P(x)$$

is equivalent to some FO-sentence, as of course it is.

Let us be precise. Suppose Q is a type $\langle 1, 1\rangle$ quantifier, say. Q is said to be *definable* in a logic L if the sentence

$$(Qx)(P_1(x), P_2(x))$$

is equivalent to some L-sentence of the same signature (in this case the signature $\{P_1, P_2\}$ consisting of two unary relation symbols). Similarly for quantifiers of other types. Now it is not hard to show

8.1 Fact. *$FO(\mathbf{Q}) \leq L$ iff each quantifier in $\mathbf{Q}$ is definable in L.*

Let us look at some examples. We saw that $FO \equiv FO(\exists_{=17})$, so clearly, for example,

(21) $$FO(Q_0) \equiv FO(Q_0, \exists_{=17}),$$

since $\exists_{=17}$, being definable in FO, is *a fortiori* definable in $FO(Q_0)$.

(22) $$FO(Q_0) \leq FO(I) \leq FO(more).$$

The first part holds since a set is infinite iff it has the same cardinality as some proper subset, so $(Q_0x)P(x)$ is equivalent to

$$(\exists x)(P(x) \,\&\, (Iy)(P(y), P(y) \,\&\, (y \neq x))).$$

The second part holds because $(Ix)(P_1(x), P_2(x))$ is clearly equivalent to

$$\sim (more\ x)(P_1(x), P_2(x)) \,\&\, \sim (more\ x)(P_2(x), P_1(x)).$$

One can show that both of the inequalities in (22) are in fact strict. (These are examples of *un*definability results; more about that in the next section.)

(23) $$FO(\mathit{most}) \leq FO(\mathit{more}),$$

since $(\mathit{most}\ x)(P_1(x), P_2(x))$ is equivalent to

$$(\mathit{more}\ x)(P_1(x) \,\&\, P_2(x)\ ,\ P_1(x) \,\&\, \sim P_2(x)).$$

Again, this is a strict inequality in general. But note that if $X \cap Y$ is a *finite* set, then we have $|X| > |Y| \Leftrightarrow |X-Y| + |X \cap Y| > |Y-X| + |X \cap Y| \Leftrightarrow |X-Y| > |Y-X|$. So when (the interpretation of) $P_1 \cap P_2$ is finite, $(\mathit{more}\ x)(P_1(x), P_2(x))$ is equivalent to

$$(\mathit{most}\ x)((P_1(x) \,\&\, \sim P_2(x)) \vee (P_2(x) \,\&\, \sim P_1(x))\ ,\ (P_1(x) \,\&\, \sim P_2(x)).$$

Let this last sentence be ψ_1. Next, when $X \cap Y$ is an *infinite* set, then $|X|$ is the maximum of $|X - Y|$ and $|X \cap Y|$, and likewise $|Y|$ is the maximum of $|Y - X|$ and $|X \cap Y|$. (These are facts of cardinal arithmetic.) It follows that, in this case, $|X| > |Y| \Leftrightarrow [|X-Y| > |Y-X|$ and $|X-Y| > |X \cap Y|]$.[1] That is, when $P_1 \cap P_2$ is infinite, $(\mathit{more}\ x)(P_1(x), P_2(x))$ is equivalent to

$$\psi_1 \ \&\ (\mathit{most}\ x)(P_1(x)\ ,\ \sim P_2(x)).$$

Let the second conjunct above be ψ_2. It now follows that, on any universe, $(\mathit{more}\ x)(P_1(x), P_2(x))$ is equivalent to

$$(\sim (Q_0 x)(P_1(x) \,\&\, P_2(x)) \ \&\ \psi_1) \ \vee\ ((Q_0 x)(P_1(x) \,\&\, P_2(x)) \,\&\, \psi_1 \,\&\, \psi_2).$$

Putting all of the above together, we have shown that

(24) $$FO(\mathit{more}) \equiv FO(Q_0, \mathit{most}).$$

All type $\langle 1 \rangle$ quantifiers are definable in terms of type $\langle 1,1 \rangle$ quantifiers (but not vice versa); in fact there is a uniform way of strenghtening a type $\langle 1 \rangle$ quantifier Q to its so-called *relativization*, which is the type $\langle 1,1 \rangle$ quantifier Q^{rel} defined by

(Rel) $Q^{\mathrm{rel}}_A XY \iff Q_X\, X \cap Y.$

[1] Proof: If $|X-Y| > |Y-X|$ and $|X-Y| > |X \cap Y|$, then $|X| \geq |X-Y| > \max(|X \cap Y|, |Y-X|) = |Y|$. On the other hand, if $|X-Y| \leq |Y-X|$, then $|X| = |X-Y| + |X \cap Y| \leq |Y-X| + |X \cap Y| = |Y|$. And if $|X-Y| \leq |X \cap Y|$, then $|X| = |X-Y| + |X \cap Y| \leq |X \cap Y| + |X \cap Y| = |X \cap Y|$ (since $X \cap Y$ is infinite) $\leq |Y|$. QED.

Roughly, Q^{rel} says (on any universe A) about X, Y what Q says on the universe X about $X \cap Y$. In other words, the quantification domain is restricted to the first argument of Q_A. We have $Q_A X \Leftrightarrow Q^{\text{rel}}_A AX$, that is, the following is logically valid:

$$(Qx)P(x) \leftrightarrow (Q^{\text{rel}}x)((x = x, P(x)),$$

which means that

(25) $$FO(Q) \leq FO(Q^{\text{rel}}).$$

Here are some examples of relativizations:

- $\forall^{\text{rel}} = \mathit{all}$
- $\exists^{\text{rel}} = \mathit{some}$
- $(\exists_{\geq n})^{\text{rel}} = \mathit{at\ least\ n}$
- $(Q_R)^{\text{rel}} = \mathit{most}$ (Q_R was defined in (14).)

So we note that the Aristotelian quantifiers are relativizations of familiar type $\langle 1 \rangle$ quantifiers. In the first three cases above, the relativizations are in turn definable from the unrelativized quantifiers, for example

$$(\mathit{all}\ x)(P_1(x), P_2(x)) \leftrightarrow (\forall x)(P_1(x) \supset P_2(x))$$

$$(\mathit{some}\ x)(P_1(x), P_2(x)) \leftrightarrow (\exists x)(P_1(x) \,\&\, P_2(x)).$$

In other words,

(26) $$FO \equiv FO(\mathit{all}) \equiv FO(\mathit{some}) \equiv FO(\exists_{\geq n}) \equiv FO(\exists^{\text{rel}}_{\geq n}).$$

However, interestingly,

(27) $$FO(Q_R) < FO(\mathit{most}).$$

Even on finite universes, in fact, saying that $X \cap Y$ has more than half the elements of X is *not* expressible in first-order logic plus the quantifier saying that a set has more than half the elements of the whole universe.

9 Undefinability

To prove that a particular quantifier Q is definable in some logic L, you need to provide a definition, that is, a defining L-sentence. This can be more or less involved (cf. the case with *more*, *most* and Q_0 in the previous section), but is often straightforward. To prove that Q is *not* so definable, however, is harder. Here you really need to verify that none of the infinitely many L-sentences works as a definition.

Sometimes one can get by with showing that L has some property that it would not have if Q were definable. This is how we saw that Q_0 is not definable in FO, using the fact that FO has the compactness property. But this is more of an exception; most logics lack compactness, or other similarly useful properties. There are, however, more elementary and direct methods of showing undefinability, but a description of these falls outside the scope of this chapter. A thorough survey of (un)definability issues for logics with monadic quantifiers is given in [Väänänen 1997].

Using these methods, it can be shown, for example, that the seemingly innocuous quantifier $most = (Q_R)^{\text{rel}}$ is essentially type $\langle 1, 1\rangle$ in a very strong sense: not only is it not definable from Q_R, but we have the following

9.1 Theorem. [Kolaitis and Väänänen 1995] most *is not definable in any logic of the form* $FO(Q_1, \ldots, Q_n)$*, where the* Q_i *are of type* $\langle 1\rangle$*. (In fact, the same holds for all the properly proportional quantifiers.)*

10 Monotonicity

Among the multitude of possibe quantifiers, the ones that actually turn up in familiar logical or linguistic contexts often have characteristic properties. Logicians want to know if logics with generalized quantifiers are well-behaved in various ways, for example if the compactness property holds for them (cf. section 7), or if they are *complete*, i.e., if their sets of logically valid sentences are recursively enumerable (can be axiomatized by a formal system). Unfortunately, many logics fail to have either of these properties; examples are $FO(Q_0)$ and $FO(most)$ (proofs of these facts can be found in [Westerståhl 1989]).

But we may more simply just look at the properties of the quantifiers themselves, and then the perhaps most conspicuous ones are the *monotonicity* properties:

- A type $\langle 1\rangle$ quantifier Q is *upward monotone*, MON↑, if for all A,

$Q_A X$ and $X \subseteq Y \subseteq A$ implies $Q_A Y$.

Downward monotonicity, MON$\downarrow$, is defined correspondingly.

- Similarly, for type $\langle 1,1 \rangle$ quantifiers we can talk about upward or downward monotonicity in the first or second argument, and we can use MON with up- or downarrows to the right and/or left to indicate this. For example, a type $\langle 1,1 \rangle$ Q is $\downarrow$MON if, for all A,

  $Q_A XY$ and $X' \subseteq X$ implies $Q_A X'Y$.

  And it is, say, $\uparrow$MON$\downarrow$ if it is upward monotone in the first argument and downward monotone in the second argument.

Now, looking at our examples we see that $\forall$, $\exists$, $\exists_{\geq n}$, Q_R, Q_0, Q_C are all MON$\uparrow$, whereas, say, $\exists_{\leq n}$ is $\downarrow$MON. A typical quantifier which is neither upward nor downward monotone is $\exists_{=n}$, but note that it is the conjunction of an upward and a downward one: $\exists_{=n} = \exists_{\geq n} \,\&\, \exists_{\leq n}$. So monotonicity is ubiquitous. Here, however, is an example of a thoroughly non-monotone type $\langle 1 \rangle$ quantifier:

$$(Q_{even})_A X \iff |X| \text{ is even.}$$

As to our type $\langle 1,1 \rangle$ quantifiers, *some* and *at least n* are $\uparrow$MON$\uparrow$, *no* is $\downarrow$MON$\downarrow$, *every* is $\downarrow$MON$\uparrow$, *more* is $\uparrow$MON$\downarrow$, and *most* is MON$\uparrow$ but, as the reader can easily verify, not monotone (in either direction) in the first argument. I is non-monotone, but as we saw in section 8 it is definable with Boolean operations from the monotone *more*. And again, a thouroughly non-monotone type $\langle 1,1 \rangle$ quantifier is *an even number of* $= (Q_{even})^{\mathrm{rel}}$.

11 Lindström quantifiers

Monadic quantifiers are, on a given universe, relations among subsets of that universe. But the business of mathematics is generalization, and it is then only natural to consider quantifiers that are relations between *relations* over the universe. This concept was introduced in [Lindström 1966], and is the official notion of a generalized quantifier in logic. Our earlier definitions easily carry over to this *polyadic* case. Let us illustrate with an example.

> A (generalized) quantifier of type $\langle 2,1,3 \rangle$ is a function Q which associates with each universe A a quantifier Q_A of type $\langle 2,1,3 \rangle$ on A, that is, a ternary relation between a binary relation over A, a subset of A, and a ternary relation over A.

Such a Q can again be seen as a variable-binding operator, such that

(Q-syn) if φ, ψ, θ are formulas, then

$$(Qxy, z, uvw)(\varphi, \psi, \theta)$$

is a formula (where all free occurrences of x, y in φ are bound by the quantifier prefix, and similarly for the other variables).

The meaning of this formula is given by the clause

(Q-sem) $A \models (Qxy, z, uvw)(\varphi, \psi, \theta)[\mathbf{a}]$ iff $(\varphi^{A,xy,\mathbf{a}}, \psi^{A,z,\mathbf{a}}, \theta^{A,uvw,\mathbf{a}}) \in Q_A$.

Here $\varphi^{A,xy,\mathbf{a}} = \{(b,c) \in A^2 : A \models \varphi[b, c, \mathbf{a}]\}$, etc. So the logic $FO(Q)$ is defined as before by adding these new clauses to the definition of a formula and of satisfaction, respectively. The reader can easily formulate all of this for the general case of a quantifier of type $\langle k_1, \ldots, k_n \rangle$.

The property ISOM is defined for such a Q in the same way as before (below for the type $\langle 2, 1, 3 \rangle$ case, so $R \subseteq A^2$, $X \subseteq A$, $S \subseteq A^3$, etc.):

(ISOM) If $(A, R, X, S) \cong (A', R', X', S')$ then $[Q_A RXS \Leftrightarrow Q_{A'} R'X'S']$.

Fact 6.1 generalizes too, so ISOM quantifiers earn the right to be called logical constants. However, they no longer say anything about quantities, so the name 'quantifier' should be taken with a grain of salt in the polyadic case. Let us look at some examples.

(28) $D_A XR \iff R$ is a dense total ordering of X (type $\langle 1, 2 \rangle$).

(29) $W_A R \iff R$ is a well-ordering of the universe (type $\langle 2 \rangle$).

To express that R is a dense total ordering of a set X is easy in FO, so $FO \equiv FO(D)$. But the notion of a well-ordering is not expressible (as can be seen by a simple application of the Compactness theorem): $FO < FO(W)$.

Let Q, Q_1, Q_2 be type $\langle 1 \rangle$ quantifiers. The next few examples illustrate so-called *lifts* of monadic quantifiers to polyadic ones; in this version they lift type $\langle 1 \rangle$ quantifiers to type $\langle 2 \rangle$ quantifiers.

(30) $Ram(Q)_A R \iff \exists X \subseteq A(Q_A X \,\&\, \forall a, b \in X(a \neq b \Rightarrow R(a, b)))$

(31) $Br(Q_1, Q_2)_A R \iff \exists X, Y \subseteq A((Q_1)_A X \,\&\, (Q_2)_A Y \,\&\, X \times Y \subseteq R)$

(32) $Res(Q)_A R \iff Q_{A^2} R$.

In all of these cases we assume that the lifted type $\langle 1 \rangle$ quantifiers are MON$\uparrow$.

The first lift is related to the statement of the so-called Ramsey Theorem (cf. any textbook of model theory).

The lift Br is an example of *branching* quantification. This idea originally stems from [Henkin 1961], who noted that the linear order of the quantifiers $\forall$ and $\exists$ in FO imposes certain restrictions that can be avoided if a *partial order* is allowed as well. This is in fact another way of generalizing FO quantification. Consider the formula

(33) $$\begin{matrix} (\forall x)(\exists y) \\ (\forall z)(\exists u) \end{matrix} \ \varphi(x, y, z, u).$$

This is read "for all x there exists y *and* for all z there exists u such that $\varphi(x, y, z, u)$", where the y depends on x but not on z, and the u depends on z but not on x. Such dependencies cannot be expressed in FO. For example, in

(34) $$(\forall x)(\exists y)(\forall z)(\exists u)\varphi(x, y, z, u)$$

u depends on x *and* z, and in

(35) $$(\forall x)(\forall z)(\exists y)(\exists u)\varphi(x, y, z, u)$$

y and u both depend on x and z. These dependencies appear clearly if we rewrite (34) and (35) by means of so-called *Skolem functions*; then (34) becomes

$$(\exists F)(\exists G)(\forall x)(\forall z)\varphi(x, F(x), z, G(x, z)),$$

and (35) is equivalent to

$$(\exists F)(\exists G)(\forall x)(\forall z)\varphi(x, F(x, z), z, G(x, z)).$$

The formula (33), on the other hand, has the intended meaning

$$(\exists F)(\exists G)(\forall x)(\forall z)\varphi(x, F(x), z, G(z)).$$

The quantifier prefix in (33) is called the *Henkin quantifier*. It can in fact be subsumed under our notion of generalized quantifier: define the type $\langle 4 \rangle$ quantifier Q^H by

$$Q^H_A R \iff \text{there are functions } f, g \text{ s.t. for all } a, b \in A, R(a, f(a), b, g(b)),$$

where $R \subseteq A^4$. Then (33) is equivalent to $(Q^H xyzu)\varphi(x, y, z, u)$. Other partially ordered quantifier prefixes with $\forall$ and $\exists$ can be defined similarly.

Adding the Henkin quantifier already extends the expressive power of FO considerably. For example, one may show that Q_0 and even *more* is definable in $FO(Q^H)$, so $FO < FO(more) \leq FO(Q^H)$.

[The proof of this observation (due to Ehrenfeucht) is too simple and too pretty to be left out here: we will express "there exists a one-one function F from P_1 to P_2"; this suffices since it means that $\sim (more\, x)(P_2(x), P_1(x))$. Consider the sentence

$$(36) \quad \begin{matrix} (\forall x)(\exists y) \\ (\forall z)(\exists u) \end{matrix} \; (((x = z) \equiv (y = u)))\,\&\,(P_1(x) \supset P_2(y)).$$

By definition, this means

$$(\exists F)(\exists G)(\forall x)(\forall z)(((x = z) \equiv (F(x) = G(z))))\,\&\,(P_1(x) \supset P_2(F(x))).$$

From the (universally quantified) first conjunct we get, first (letting z be x) that $F(x) = G(x)$ for all x, so $F = G$, and second, that if $x \neq z$ then $F(x) \neq F(z)$, so F is one-one, and we are done!]

Now [Barwise 1979] suggested that one may also consider branching of (certain) other quantifiers than $\forall$ and $\exists$, and the polyadic $Br(Q_1, Q_2)$ is an example of this, which we could emphasize by writing the formula $(Br(Q_1, Q_2)xy)\varphi(x, y)$ as

$$\begin{matrix} (Q_1 x) \\ (Q_2 y) \end{matrix} \; \varphi(x, y).$$

It thus says that there is a set X satisfying Q_1 and a set Y satisfying Q_2 such any pair (x, y) with $x \in X$ and $y \in Y$ satisfies $\varphi(x, y)$. The 'order-independence' of the lifted quantifiers here is witnessed by the fact that the formula is equivalent to

$$\begin{matrix} (Q_2 y) \\ (Q_1 x) \end{matrix} \; \varphi(x, y).$$

So in fact we have two ways of generalizing FO quantification: one is through the concept of a (Lindström) generalized quantifier (which, as we noted, essentially occurs already with Frege), and the other is through relaxing the linear left-right order of FO logic. As we saw, the latter can, for the case of $\forall$ and $\exists$, be subsumed under the former. But there also arises the question as to whether we can 'branch' arbitrary generalized quantifiers. Barwise considered some cases of branching of MON$\uparrow$ quantifiers, but he explicitly stated that another definition is required for the branching of

MON↓ quantifiers, and he also claimed that the branching of, say, a MON↑ and a MON↓ quantifier "makes no sense". In spite of this, others have tried to express the meaning of arbitrary partially ordered prefixes of arbitrary generalized quantifiers, cf. [Sher 1997]. It remains to be seen, in my opinion, whether these ideas yield a fruitful notion of (generalized) quantifier.

The lift $Res(Q)$, finally, is called the *resumption* (sometimes *vectorization*) of Q. Looking at a binary relation R as a set of ordered pairs, $Res(Q)_A R$ simply says about R what Q says about that set of pairs. For example,

$$Res(\exists_{\geq n})_A(R) \iff |R| \geq n,$$

i.e., $Res(\exists_{\geq n})_A(R)$ says that R has at least n pairs. Likewise,

$$Res(Q_R)_A(R) \iff |R| > |A^2 - R|.$$

As one would expect, polyadic quantifiers have in general more expressive power than monadic ones. As to the lifts, one can, for example, show that $(Br(Q_1, Q_2)xy)P(x, y)$ is usually stronger than the 'linear versions' $(Q_1x)(Q_2y)P(x, y)$ and $(Q_2y)(Q_1x)P(x, y)$. Indeed, [Hella, Väänänen and Westerståhl 1997] proves that these lifts are *essentially* polyadic:

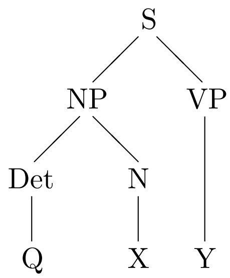
11.1 Theorem. *$Br(Q_R, Q_R)$ is not definable in any logic of the form $FO(Q_1, \ldots, Q_n)$ where the Q_i are monadic, and the same holds for $Ram(Q_R)$.*

Undefinability results for polyadic quantifiers can be very hard to prove. An example is the result in [Luosto 1996] that $Res(Q_R)$ too is not definable from any finite number of monadic quantifiers added to FO; this proof requires quite advanced combinatorics.

12 Quantifiers and natural language

The most obvious connection between (generalized) quantifiers and natural languages is that many of these languages have a fundamental sentence construction of the form $[[[\mathrm{Q}]_{Det}][\mathrm{X}]_N]]_{NP}][\mathrm{Y}]_{VP}]_S$, or, in diagrammatic form,

(37)



That is, (declarative) sentences are often formed by a Noun Phrase and a Verb Phrase, where the Noun Phrase consists of a Determiner and a Noun.[2] Now, both nouns (*man*, *teacher*, *hungry dog*, *student who likes a teacher*, ...) and verb phrases (*runs*, *loved Billy*, *gave a flower to some shop owner*, ...) are naturally interpreted as sets, i.e., as subsets of the universe of discourse. Therefore, the determiner (*every*, *no*, *most*, *at least three*, *several of John's ten*) can be taken as a relation between sets, that is, as a type $\langle 1, 1\rangle$ quantifier (on the universe, but the Det gives a quantifier on each universe, so it corresponds to a generalized quantifier in our sense). For example,

(38) No student likes Henry

(39) All but three teachers smoke

(40) Most yellow cats are friendly

(41) Two thirds of John's friends are linguists

etc.

Other types as well turn up in connection with natural languages. The expressions *everything* and *something* naturally correspond to the type $\langle 1\rangle$ $\forall$ and $\exists$. More generally, NP's may be interpreted as type $\langle 1\rangle$ quantifiers, so that, for example, *most students* denotes the set of subsets X of the universe whose intersection with the sets of students contains more than half of the students, and *all but three dogs* denotes the set of those X such that the complement of X with respect to the set of dogs has exactly three elements, etc. We also saw that proper names like *John* can be taken as type $\langle 1\rangle$

[2] All of these phrases may in turn have internal structure; in particular, noun phrases can occur in many different positions in a sentence. Also, quantification can be effected by other means than determiners, for example using adverbs — I am just looking at the simplest case here.

quantifiers (note that this example as well as the last two do not satisfy ISOM).

In the sentence

(42) More students than teachers smoke,

we may see the type $\langle 1,1,1\rangle$ *more than* ((4) from section 4) at work. But also polyadic lifts appear in the context of natural language quantification; a survey of this can be found in [Keenan and Westerståhl 1997]. I will not go further into these matters here, but end with a few more words about the central type $\langle 1,1\rangle$ case, i.e., the determiner denotations.

Given the vast number of mathematically possible type $\langle 1,1\rangle$ quantifiers, a reasonable question is whether there are constraints as to which of these can be realized in natural languages. A prime observation is that the noun argument X in (37) plays a special role: it *restricts the domain of quantification*. This is borne out by looking at actual examples:

(43) Exactly three dogs barked

can be seen as quantifying over the subuniverse of dogs; the non-dogs of the universe are irrelevant for the truth or falsity of this sentence. Also, a special role of the noun argument is consistent with the syntactic structure of (37). An early observation (in [Barwise and Cooper 1981] and [Keenan and Stavi 1986]) was that determiner denotations are *conservative*: they satisfy

(CONS) $Q_A XY \Leftrightarrow Q_A X\, X\cap Y$, for all A and all $X, Y \subseteq A$.

This means, in effect, that the part $Y-X$ in Figure 1 (section 5) plays no role in the truth conditions of $Q_A XY$. This seems to hold for determiner denotations, but it does *not* hold, for example, for the otherwise mathematically perfectly natural quantifiers I and *more* (section 4). And indeed, there do not seem to be any determiner expressions in natural languages which denote these quantifiers.

There is one more aspect of domain restriction, however: the part $A-(X\cup Y)$ should not matter to the truth conditions either. This can be expressed as the following condition of *extension*, first proposed by van Benthem (cf. [van Benthem 1986]):

(EXT) If $X, Y \subseteq A \subseteq A'$, then $Q_A XY \Leftrightarrow Q_{A'} XY$.

That is, what a determiner denotes on a given universe does not 'change' if we go to a larger universe. So, for example, there could not be a determiner *blik*, say, which meant *some* on universes with less than ten elements, but *most* on larger universes (note that this quantifier would still be conservative).

Now recall that the idea of domain or universe restriction was already defined in section 8 in terms of the notion of relativization. And indeed, conservativity and extension together capture exactly the same idea:

12.1 Fact. *A type $\langle 1,1\rangle$ quantifier satisfies CONS and EXT iff it is the relativization of some type $\langle 1\rangle$ quantifier.*

To see this, check first that Q^{rel} always satisfies CONS and EXT. In the other direction, any CONS and EXT Q' has a type $\langle 1\rangle$ 'counterpart' Q defined by $Q_A X \Leftrightarrow Q'_A AX$: then $Q^{\mathrm{rel}}_A XY \Leftrightarrow Q_X X \cap Y$ (by (Rel) in section 8) $\Leftrightarrow Q'_X X\ X \cap Y$ (by definition) $\Leftrightarrow Q'_A X\ X \cap Y$ (by EXT) $\Leftrightarrow Q'_A XY$ (by CONS), so $Q' = Q^{\mathrm{rel}}$.

So quantifiers that are denoted by determiners in natural languages satisfy CONS and EXT. They also satisfy ISOM. The ISOM+CONS+EXT quantifiers form a natural class, but there have been several attempts to formulate further constraints or 'linguistic universals' that single out (important subclasses of) the 'natural language quantifiers'. Prime examples here are the various monotonicity properties discussed in section 10. It may seem — and it has been suggested — that all (monadic) quantifiers occurring in natural languages are Boolean combinations of monotone ones. However, an apparent exception to this would be *an even number of* $= (Q_{even})^{\mathrm{rel}}$. And it is true that one can show (this follows from a result in [Väänänen 1997]) that Q_{even} is not definable from MON↑ type $\langle 1\rangle$ quantifiers. Perhaps surprisingly, however, it *is* definable from the relativization of such a quantifier, in fact from a CONS, EXT, ISOM, and MON↑ type $\langle 1,1\rangle$ quantifier. Let us see how. We have

$$\textit{an even number of}_A XY \iff |X \cap Y| \text{ is even.}$$

Now define Q by

$$Q_A XY \iff \begin{cases} |X \cap Y| \geq 1 & \text{if } |X| \text{ is even} \\ |X \cap Y| \geq 2 & \text{if } |X| \text{ is odd.} \end{cases}$$

Q is clearly CONS, EXT, ISOM, and upward monotone in the *second* argument. Notice then that if $a \in X$,

$$Q_A X\{a\} \iff |X| \text{ is even.}$$

But then we have $|X \cap Y|$ is even $\Leftrightarrow X \cap Y = \emptyset$ or $\exists a \in X \cap Y\ Q_A X \cap Y\{a\}$. That is, (*an even number of x*)$(P_1(x), P_2(x))$ is equivalent to

$$\sim \exists x(P_1(x) \,\&\, P_2(x)) \vee \exists x(P_1(x) \,\&\, P_2(x) \,\&\, (Qy)((P_1(y) \,\&\, P_2(y)), (y = x))).$$

Perhaps one could still argue that Q is 'unnatural' in some sense, but I will leave the matter here.

In early days of linguistic semantics it was sometimes suggested that first-order logic, FO, suffices for the formalization of natural languages. This thesis can be refuted in many ways, I think, but perhaps the most convincing rebuttal comes from the theory of quantifiers. Certainly *most* is a natural language quantifier, but we have seen that, even if one restricts attention to finite universes, it is not FO definable, indeed it is not definable from any type $\langle 1 \rangle$ quantifiers (Theorem 9.1). Essentially stronger logics than FO are needed to capture the intricacies of quantification in natural languages.

13 Suggested further reading

A detailed exposition of most of the aspects of quantification touched on in this chapter can be found in [Westerståhl 1989]. A more recent survey article, emphasizing the conncection with natural languages, and in particular the occurrence of polyadic lifts, is [Keenan and Westerståhl 1997]. There are several technical papers on the expressive power of various quantifiers; I would suggest [Kolaitis and Väänänen 1995], [Väänänen 1997], and [Hella, Väänänen and Westerståhl 1997], where the details are spelled out in an accessible way. The canonical collection of mathematical papers on logics with generalized quantifiers, and more generally on logics defined in a model-theoretic way, is [Barwise and Feferman 1985]. A more philosophical approach to the logic of quantifiers can be found in several of the papers in the collection [van Benthem 1986]. All of the work cited so far approaches quantification from a logical point of view. For those interested in the various forms that quantification can take in the world's languages, [Bach et al. 1995] is an invaluable source. The connection between this more empirical work and the logic of quantification still remains to be fully explored.

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