Consequence Mining *

Constants versus Consequence Relations

Denis Bonnay
Université Paris Ouest Nanterre
IREPH (U. Paris Ouest) and IHPST (CNRS / Paris I / ENS)

Dag Westerståhl
University of Gothenburg and Stockholm University

Abstract. The standard semantic definition of consequence with respect to a selected set X of symbols, in terms of truth preservation under replacement (Bolzano) or reinterpretation (Tarski) of symbols *outside* X, yields a function mapping X to a consequence relation $\Rightarrow_X$. We investigate a function going in the other direction, thus *extracting* the constants of a given consequence relation, and we show that this function (a) retrieves the usual logical constants from the usual logical consequence relations, and (b) is an inverse to — more precisely, forms a Galois connection with — the Bolzano-Tarski function.

Keywords: Consequence relation, Constant, Logical Constant, Bolzano, Tarski, Galois connection, Replacement, Substitution

1. Introduction

The close connection between logical constants and logical consequence is known to every logic student. After selecting a suitable set of logical constants, a relation of logical consequence is defined, either semantically via truth-preservation or syntactically via rules of derivation. Probably less noticed is the fact that the semantic definition of consequence allows *any* set of symbols or words in the language to be chosen as constants. In fact, Bolzano — the first to systematically study consequence — insisted on this generality, and it holds equally for the Tarskian model-theoretic version. The selected symbols are *constants* precisely in the sense that their meaning is held fixed. Whether they are *logical* or not is a further issue. The semantic definition simply provides a function from arbitrary sets of symbols to consequence relations.

The aim of this paper is to study a function in the opposite direction — *extracting* or *mining* constants from consequence relations — which is an inverse, in a suitable sense, to the Bolzano-Tarski function. It

* We thank Johan van Benthem, Stephen Read, and Göran Sundholm for helpful remarks, and in particular Lloyd Humberstone, Dave Ripley, and an anonymous referee for very useful comments on an earlier version of this paper. We also thank the audiences at several seminars and workshops where we presented this material, in various stages, for inspiring discussion.

seems rather natural to inquire whether such a function exists, and the answer turns out to be not quite trivial. But there is another motivation. Speakers in many cases have rather strong intuitive opinions about 'what follows from what'. Such judgments, which can be taken to partially express a consequence relation, make it clear that certain words, but not others, are constant in the sense that *replacing* them by other symbols (of the same category) can destroy the validity of an inference. These judgments appear quite basic; they do not require a logic education or theoretical reflection on grammar. Our extraction method relies only on facts of validity or non-validity of particular inferences. It applies to any consequence relation, logical or not. Thus, it provides a new method of isolating the constants of a language.

1.1. Kinds of constants

The recognition that different words play different roles for the consequence relation is an old one. There is the medieval distinction between *categorematic* and *syncategorematic* terms, the dominant idea being that the latter had no independent meaning (did not correspond to anything 'definite' in the mind).[1] The linguistic distinction between *functional* and *descriptive* terms is (partly) a modern analogue. When Tarski (1936) gave the first version of the model-theoretic definition of *logical* consequence, he recognized that one needs grounds for selecting the logical constants — and that he had none. So did Bolzano, one hundred years earlier.[2] Such grounds have subsequently been given, notably in terms of *invariance* under (a group of) transformations.[3]

In the present paper, however, we avoid these difficult and controversial demarcation problems by starting at the other end: taking

[1] This particular idea may be hard to defend in present day semantics, where familiar logical constants like *every* and *or* do receive seemingly independent interpretations of their own.

[2] It is only in one short paragraph in §148 of Bolzano (1837) that he mentions the problem and connects it to defining what he calls logical analyticity. According to Bar-Hillel (1950), this was "surely one of the most important and ingenious single logical achievements of all times." (p. 101) Bar-Hillel notes that this notion of logical truth or analyticity is not used in the rest of the book, and hypothesizes that it was a late insight, inserted just before publication. Be that as it may, an equally important achievement in *Wissenschaftslehre*, which pervades the book, is the analysis of *consequence* in terms of an arbitrary selection of constants.

[3] Starting with Tarski (1986), the aim is to characterize logical notions as the ones invariant under the most general kind of transformation. This field has been rather active recently; see, for example, Bonnay (2008) and Feferman (2010). For a non-technical overview of various approaches to logical constants, see MacFarlane (2009).

consequence relations as primitive we extract their constants.[4] The set of thus extracted constants is relative to a consequence relation. For familiar such relations, one expects it to include the familiar logical constants, but it may contain more, for example, if analytic consequences are admitted. It depends on the consequence relation.

1.2. Bolzano reversed

We already indicated the almost embarrassingly simple idea on which the extraction method is based. In a given valid inference, it may be perfectly clear which words or symbols are constant and which are not: Just replace them by others of the same category and see what happens. If validity (according to the given consequence relation) is always preserved, we are not dealing with a constant. But if validity can be destroyed in this way, we are. In this straightforward way, the constants of that inference are the words or symbols *essential* to its validity. Here is an example.

(1) a. Most French movies encourage introspection
 b. All movies which encourage introspection are commercial failures
 c. *Hence:* Most French movies are commercial failures

(1) is presumably valid according to most natural notions of consequence: the conclusion *follows from* the premises. Now replace words like *French*, *movies*, etc. with others of the same category (in some suitable sense):

(2) a. Most red sports cars are convertibles
 b. All cars which are convertibles are unsuitable for cold climates
 c. *Hence:* Most red sports cars are unsuitable for cold climates

Nothing happens. The inference is still valid. In a sense it is the *same* inference. But try instead to replace words like *most* or *all*:

(3) a. No French movies encourage introspection
 b. All movies which encourage introspection are commercial failures
 c. *Hence:* No French movies are commercial failures

[4] The only precursor to the idea of defining constants in terms of validity that we know of is Carnap (1937). Carnap suggested that the set of logical and mathematical constants is the maximal set of expressions such that every sentence built out of these expressions is either valid or invalid. Our proposal is essentially different, and closer in spirit to standard accounts of validity.

This is not only invalid: whether it is valid or not seems to have nothing to do with the validity of (1)! Hence, *most* is a constant in (1), *French* is not.

These observations are so obvious that they are easy to overlook. But that doesn't make them trivial. The method of replacing words by others of the same category, and seeing what happens, was precisely Bolzano's approach to consequence. In his case, the criterion was that *truth* should be preserved (for consequence to hold). In the reverse direction, we ask instead if *validity* is preserved.

1.3. Extraction

So far, however, we have only given a way to detect constanthood relative to a particular valid inference. Now we need to extract the constants from the consequence relation itself. A first proposal was made in Peters and Westerståhl (2006), Ch. 9, where it was suggested that a symbol u is constant if *every* valid inferences in which it occurs essentially can be destroyed by some replacement. The problem with this idea is that the qualification "in which it occurs essentially" is crucial, and must be explained independently, as the following examples from first-order logic illustrate:

(4) a. $Pa \models Pa \vee \exists xRx$

b. $\exists xPx, \forall x(Px \leftrightarrow Rx) \models \exists xRx$

In both of these inferences, the quantifier $\exists$ can be replaced by any type $\langle 1 \rangle$ quantifier Q, i.e. with any symbol of the same category, without destroying validity. Cases like (4-a) are easy to set aside: the occurrence of $\exists$ there is *spurious*, in that (4-a) is an instance of a more general inference (i.e. $\varphi \models \varphi \vee \psi$) in which $\exists$ does not occur. But (4-b) is more tricky: it expresses a principle of *extensionality*, and it is not clear how all such principles can be set aside on syntactic or other grounds. In particular, the suggested notion of constanthood makes it a non-trivial task to verify that the usual logical symbols in familiar consequence relations, such as first-order consequence, are indeed constants according to the criterion.

Here we shall rely on a weaker idea: It suffices, for a word or symbol to be constant relative to the given consequence relation, that there be *at least one* valid inference which can be destroyed by a replacement. We will see that this definition satisfies the following two adequacy criteria:

- It yields the expected constants when applied to familiar notions of logical consequence.

– It can provide an inverse, in a precise sense, to Bolzano's function from sets of symbols to consequence relations.

The second criterion turns out to be quite fruitful. We will see what kind of inverse relationship can be expected, and get to know in some detail under what conditions on the language and on the notion of consequence it holds. In fact, most of the technical work to follow was driven by the goal of understanding the relationship between the two functions.

1.4. Plan

We will draw an abstract mathematical picture of the situation just indicated for a given language, with the two functions, one forming consequence relations from constants, and the other extracting constants from consequence relations. While making as few assumptions as possible about the language, we shall start from Bolzano's substitutional perspective rather than the model-theoretic one: there is a given *interpreted* language, in which meaningful expressions can be *replaced*, rather than *reinterpreted*. Our reason for doing so is not historical however. First, we want to start with a language for which a consequence relation exists: such a language is an interpreted language and the consequence relation corresponds to intuitions speakers have about the validity of inferences. Second, our test for extraction is substitutional: it has to be, if it is meant to rely only on speakers' judgments about validities. A substitutional, rather than semantic, definition of consequence is the natural match for such a test. We will eventually show in Section 8 how to cover the semantic definition of consequence as well.

As a bonus, starting from Bolzano we get a firmer grip on the ways in which lack of expressivity in the language may affect consequence relations, for example, by inability to express an existing counter-example to an inference. It turns out that this well-known problem for substitutional accounts of consequence is *not* what prevents our extraction function from being an inverse in the sense we want. Rather, it is another, less noticed feature, namely that when you add new symbols to the language, valid inferences in the original language may cease to be valid. In other words, the language need not be *conservative over expansions*, in contrast with the situation for model-theoretically defined consequence. We will see that under suitably strong assumptions about the richness of the language or, alternatively, about the possibility of expanding it, the required inverse relationship does hold.

Thus we begin, after precise definitions of the notions of *language*, *replacement*, and *consequence relation* (Section 2), with a precise account of Bolzano's function from sets of symbols to consequence relations, and some of its properties (Sections 3 and 4). Section 5 introduces the extraction function in the other direction, and looks at its behavior on certain examples, compared to how we would like it to behave, in view of our two adequacy criteria. It is the second criterion that requires additional assumptions, and the rest of the paper investigates the effects of these. The form of inverse relationship that we find is that of a *Galois connection* (Sections 6 and 7). As just indicated, it holds under certain richness requirements, and our most general version of the substitutional framework actually subsumes the model-theoretic one (Section 8). Section 9 contains concluding remarks and directions for further study.

2. Preliminaries

2.1. Languages

In the Bolzano setting with an interpreted language, we shall take every sentence to be either *true* or *false*. We need very few assumptions about what sentences look like or how they are structured. Most of what we need is captured in the following definition.

DEFINITION 1. *An (interpreted)* language *is a triple*

$$L = \langle Symb_L, Sent_L, Tr_L \rangle,$$

where

(i) $Symb_L$ *is a countable set of atomic* symbols*;*
(ii) $Sent_L$ *is a set of* sentences*, which are finite strings of signs, some of which belong to* $Symb_L$*;*[5]
(iii) Tr_L*, the set of* true *sentences, is a subset of* $Sent_L$*.*

As long as we only consider replacement of symbols by other symbols, we may disregard finer aspects of the structure of sentences, such as tree structure. But since we cannot realistically expect a symbol to be meaningfully replaceable by *any* other symbol, we shall presuppose a partition of symbols into categories. More precisely, let a set *Cat* of *categories* be given. Given a language L and a category C in *Cat*, we write C_L for the set of symbols of L that are of category C. We assume

[5] Think of the other signs as grammatical morphemes, parentheses, commas, variables, etc. We assume there are at most countably many of those too, and then $Sent_L$ is also countable.

that for each language L,

$$Cat_L = \{C_L : C \in Cat\}$$

is a *partition* of $Symb_L$. Note that symbols in distinct languages can be of the same category.

Whenever convenient, we drop the subscript $_L$. We let $u, v, u', \ldots$ vary over *Symb*, $\varphi, \psi, \ldots$ over *Sent*, and $\Gamma, \Delta, \ldots$ over subsets sets of *Sent*. Also, V_φ is the set of symbols *occurring* in φ. Likewise, $V_\Gamma = \cup\{V_\varphi : \varphi \in \Gamma\}$.

2.2. Replacement

A *replacement* is a partial function ρ from *Symb* to *Symb* that respects categories: if $u \in dom(\rho)$ is of category $C \in Cat_L$, then so is $\rho(u)$. We write $\varphi[\rho]$ for the result of replacing each occurrence of u in φ by $\rho(u)$.[6] It is convenient to assume that $V_\varphi \subseteq dom(\rho)$ — in words, ρ is a replacement *for* φ — so that ρ is the identity on symbols that don't get replaced.

We make the extra assumption that *Sent* is *closed* under replacement. Then the following conditions hold:[7]

(5)
a. If ρ is a replacement for φ, $\varphi[\rho] \in Sent$ and $V_{\varphi[\rho]} = range(\rho \restriction V_\varphi)$
b. $\varphi[id_{V_\varphi}] = \varphi$
c. If ρ, σ agree on V_φ, then $\varphi[\rho] = \varphi[\sigma]$.
d. $\varphi[\rho][\sigma] = \varphi[\sigma\rho]$, when σ is a replacement for $\varphi[\rho]$

2.3. Consequence relations

We take consequence relations to hold between sets of sentences and sentences. This agrees with Bolzano, and with much of the approach to consequence taken in the early 20th century by Hertz (1923), Lewis and Langford (1932), Tarski (1930a; 1930b), and Gentzen (1932).[8] The same holds for the following definition.

[6] We often write $\varphi[u_1/u'_1, \ldots, u_n/u'_n]$ instead $\varphi[\rho]$, when $\rho(u_i) = u'_i$ and ρ is the identity on all other symbols in φ. Also, we only replace symbols by symbols, whereas it would seem more natural to replace symbols by other expressions of the same category. For the points we wish to make in this paper, however, this extra generality is not necessary.

[7] These are essentially the conditions in Peter Aczel's notion of a *replacement system* from (Aczel, 1990).

[8] Other notions take the conclusion to be a set of sentences as well, or use sequences or multisets instead of sets. For a comparison of early notions of consequence, see Shoesmith and Smiley (1978), especially Section 1.1.

DEFINITION 2.

(a) A relation $R \subseteq \wp(Sent_L) \times Sent_L$ *is a* consequence relation *in* L *iff it satisfies*

(R') If $\varphi \in \Gamma$*, then* $\Gamma\, R\, \varphi$*.*

(T) If $\Delta\, R\, \varphi$ *and* $\Gamma\, R\, \psi$ *for all* $\psi \in \Delta$*, then* $\Gamma\, R\, \varphi$*.*

(b) $CONS_L$ *is the set of consequence relations in* L *which are* truth-preserving*: whenever* $\Gamma\, R\, \varphi$ *and (every sentence in)* Γ *is true,* φ *is also true.*

We let $\Rightarrow, \Rightarrow', \ldots$ vary over consequence relations. (T) for *transitivity* and (R') stands for a generalized version of *reflexivity*,

(R) $\varphi \Rightarrow \varphi$ for all $\varphi \in Sent_L$

It easily follows that consequence relations are also *monotone*:

(M) If $\Gamma \Rightarrow \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \Rightarrow \varphi$.[9]

If only finite sets of premises are considered, we say that $\Rightarrow$ is *finitary*. Results using the finiteness restriction will be marked (FIN). A weaker constraint is to consider *compact* consequence relations, in the sense that

If $\Gamma \Rightarrow \varphi$, then $\Gamma' \Rightarrow \varphi$ for some finite subset Γ' of Γ.

Define:

(6) a. $\Gamma \Rightarrow^{max} \varphi$ iff it is not the case that Γ is true and φ is false.

b. $\Gamma \Rightarrow^{min} \varphi$ iff $\varphi \in \Gamma$.

$\Rightarrow^{max}$ is essentially *material implication.* $\Rightarrow^{max}$ and $\Rightarrow^{min}$ are the smallest and the largest elements of the partial order $(CONS_L, \subseteq)$. Note

---

[9] Indeed, (R')+(T) is equivalent to the often used combination (R)+(M)+(CS), where (CS) is *cut for sets*:

(CS) If $\Gamma \cup \Delta \Rightarrow \varphi$ and $\Gamma \Rightarrow \psi$ for all $\psi \in \Delta$, then $\Gamma \Rightarrow \varphi$.

As to truth preservation, let a *valuation* be a function v from *Sent* to $\{\mathrm{T}, \mathrm{F}\}$. Truth preservation for v is defined in the obvious way, and to each set K of valuations corresponds the consequence relation $\vdash_K = \{(\Gamma, \varphi) : (\Gamma, \varphi)$ preserves truth for each $v \in K\}$. Then one can show that each consequence relation is of the form $\vdash_K$ for some K (see (Shoesmith and Smiley, 1978), Section 1.1, for a proof), which gives some motivation for the chosen defining conditions of consequence relations. In our setting here with an interpreted language L, we have chosen a fixed valuation (the 'intended' one).

also that for every truth-preserving relation R on $\wp(Sent_L) \times Sent_L$, there is a smallest consequence relation $cl_L(R) \in CONS_L$ extending R. $cl_L(R)$ is the intersection of all consequence relations in which R is included.

3. Consequence from constants

Of particular interest are the consequence relations *generated* from a set of constants. The idea is familiar to every logician: φ follows from Γ, relative to a set X of constants, iff any reinterpretation of symbols *outside* X that makes the premises true also makes the conclusion true. In the Bolzano setting with an interpreted language, however, we do not reinterpret symbols but *replace* them.

3.1. Bolzano consequence

Bolzano stressed the fact that we are in principle free to regard *any* set of symbols as constants. As pointed out in van Benthem (2003), one may thus think of Bolzano consequence as a *ternary* relation, between a set of premises, a conclusion, and a set X of symbols treated as constants. Equivalently, we shall define a *function* $\Rightarrow_{\_}$ from sets of symbols to consequence relations, as follows:[10]

DEFINITION 3. *For any $X \subseteq Symb_L$, define the relation $\Rightarrow_X$ by*

> $\Gamma \Rightarrow_X \varphi$ *iff for every replacement ρ that is the identity on X, if $\Gamma[\rho]$ is true, so is $\varphi[\rho]$.*

A relation of the form $\Rightarrow_X$ is called a *Bolzano consequence* (relation), and we let $BCONS_L$ be the set of Bolzano consequences in L. It is straightforward to verify the following claims.

FACT 4.

(a) $BCONS_L \subseteq CONS_L$

(b) *In addition, Bolzano consequences are* base monotone*, in that*

[10] We do not follow Bolzano to the letter; for example, we do not require, as he did, that the set of premises should be *consistent* in order to have any consequences. For a discussion of this and several other aspects of Bolzano's notion of consequence, see van Benthem (2003). Another departure from Bolzano's original approach is that ours is syntactic, replacing *symbols*, whereas he replaced *concepts* ('Vorstellungen an sich'). Incidentally, this might make his account less vulnerable to detrimental effects due to lack of symbols in the language.

$X \subseteq Y$ implies $\Rightarrow_X \subseteq \Rightarrow_Y$

(c) *$(BCONS_L, \subseteq)$ is a partial order which has $\Rightarrow_\emptyset$ as its smallest and $\Rightarrow_{Symb}$ as its largest element.*

So $(BCONS_L, \subseteq)$ is a sub-order of $(CONS_L, \subseteq)$, and we see that

(7) $\quad \Rightarrow^{max} \; = \; \Rightarrow_{Symb}$

It often happens,[11] however, that

(8) $\quad \Rightarrow^{min} \; \subsetneq \; \Rightarrow_\emptyset$

so $BCONS_L$ can be a proper subset of $CONS_L$. The following is trivial but fundamental:

LEMMA 5. (Replacement Lemma) *If $\Gamma \Rightarrow_X \varphi$ and ρ is the identity on X, then $\Gamma[\rho] \Rightarrow_X \varphi[\rho]$.*

Proof. Use the composition property (5-d) in Section 2.2 of replacement, noting that if both ρ and σ are the identity on X, so does $\sigma\rho$. □

Furthermore, from base monotonicity and (5-c) we see that only symbols occurring in premises and conclusion matter for Bolzano consequence:

LEMMA 6. (Occurrence Lemma) $\Gamma \Rightarrow_X \varphi$ *iff* $\Gamma \Rightarrow_{X \cap V_{\Gamma \cup \{\varphi\}}} \varphi$.

3.2. Example: propositional logic

Let PL be a standard language of *propositional logic*, whose symbols consist of a suitable set of connectives and an infinite supply of propositional letters — say, $Symb_{PL} = \{\neg, \vee, \wedge\} \cup \{p_0, p_1, \ldots\}$ with the (non-empty) categories 'unary truth function', 'binary truth function', and 'propositional letter' — and let $\models_{PL}$ be the corresponding (classical) consequence relation. The usual definition of consequence in this language is model-theoretic, but we can 'simulate' $\models_{PL}$ also in our substitutional setting, where $p_0, p_1, \ldots$ are sentences with fixed truth

[11] For example, in propositional logic, $***p \Rightarrow_\emptyset *p$, where p is a propositional letter and $*$ a unary truth function. Likewise, in all of the particular examples to follow, $\Rightarrow_\emptyset$ is distinct from $\Rightarrow^{min}$.

values, and the truth values of complex sentences are computed from these by the usual truth tables. Replacing proposition letters by others amounts to 'assigning' arbitrary truth values to them, under a simple assumption: let us say that PL, viewed as an interpreted language, is *non-trivial* iff the sequence of truth values of $p_0, p_1, \ldots$ is not eventually constant. Clearly,

(9) If PL is non-trivial, then for every (countable) sequence $\alpha_1, \alpha_2, \ldots$ of truth values there are propositional letters $p_{i_1}, p_{i_2}, \ldots$ such that the truth value of p_{i_j} is α_j, for all j.

Using (9), one easily verifies:

FACT 7. *If PL is non-trivial, then $\Gamma \models_{PL} \varphi$ iff $\Gamma \Rightarrow_{\{\neg,\vee,\wedge\}} \varphi$.*

3.3. Example: first-order logic

Now let FO be a standard language of *first-order logic*. Setting parentheses and variables aside,[12] we take

$$Symb_{FO} = \{\neg, \vee, \wedge, \forall, \exists, =\} \cup \{P_0, P_1, \ldots, c_0.c_1, \ldots\}$$

with obvious categories such as 'type $\langle 1 \rangle$ quantifier', 'binary predicate symbol', 'individual constant', etc.

Let $\models_{FO}$ be the standard consequence relation for FO. Once an interpretation is fixed for all symbols (including the non-logical ones), all sentences in FO get a truth-value and Bolzano's definition of consequence applies. The following inclusion is straightforward:

(10) $\models_{FO} \quad \subseteq \quad \Rightarrow_{\{\neg,\wedge,\vee,\forall,\exists,=\}}$

However, the converse inclusion is not true for all languages FO – this was precisely the motivation in Tarski (1936) for giving a semantic and not a substitutional definition of logical consequence. For example, if every name a in the language is such that for all predicates P, either Pa is false or $\forall x Px$ is true, then we get $Pa \Rightarrow_{\{\forall\}} \forall x Px$ for an arbitrary predicate P. As we shall see in section 8, one way to simulate $\models_{FO}$ is to consider expansions of the base language, in order to alleviate the limitations of the substitutional account.[13]

At first blush, one might think the Bolzano approach simply amounts to FO with *substitutional interpretation of the quantifiers*, but this is

[12] This in keeping with Definition 1, since it was not required that $Symb_L$ contains all symbols occurring in sentences of the language (see also footnote 5).

[13] The semantic definition to be used in section 8 will not exactly be the classical $\models_{FO}$ since we will not consider domain variations.

not so. The reason is that in standard definitions of logical consequence with substitutional quantification, as in Dunn and Belnap (1968), only the quantifiers are interpreted substitutionally, but not the rest of the language. In more detail, in their substitutional account of FO, call it FO-subst, truth is defined relative to an arbitrary assignment ν of truth values to the atomic sentences, extended in the usual way to negations and conjunctions, and to universally quantified sentences by

(11) $\forall x\varphi(x)$ is true relative to ν iff $\varphi(c)$ is true relative to ν for all individual constants c.

Logical consequence is defined as follows:

(12) $\Gamma \models_{FO\text{-subst}} \varphi$ iff every assignment ν relative to which Γ is true is also one relative to which φ is true.

If there are infinitely many names, Dunn and Belnap show that $\models_{FO\text{-subst}}$ and $\models_{FO}$ agree when only finitely many premises are considered. As a consequence, the previous counterexample to the inclusion of $\Rightarrow_{\{\neg,\wedge,\vee,\forall,\exists,=\}}$ in $\models_{FO}$ carries over to $\models_{FO\text{-subst}}$.

When infinite sets of premises are considered, $\models_{FO\text{-subst}}$ and $\models_{FO}$ cease to agree, as witnessed by the following:

$$\{Pc\colon c \text{ is a name}\} \models_{FO\text{-subst}} \forall x Px$$

Now there will be interpreted languages FO such that this is not valid according to $\Rightarrow_{\{\neg,\wedge,\vee,\forall,\exists,=\}}$, because, for example, all names can be replaced by a given name d such that Pd is true, even though $\forall xPx$ is false. Therefore, $\models_{FO\text{-subst}}$ and $\Rightarrow_{\{\neg,\wedge,\vee,\forall,\exists,=\}}$ are incomparable, in the sense that it is possible to find interpreted languages FO such that the former is not included in the latter, or the other way around.[14]

3.4. Two toy languages

We now describe in some detail two very simple languages and their consequence relations. These examples, and variants of them, will serve later on to illustrate various features of Bolzano consequence.

3.4.1. *The language L_1*

Let the language L_1 be specified as follows:

$$Symb_{L_1} = \{R, a, b\} \quad \text{(with } a, b \text{ of the same category)}$$

[14] Thanks to an anonymous referee for pointing this out, and correcting an earlier mistake of ours.

$$Sent_{L_1} = \{Raa, Rab, Rba, Rbb\}$$
$$Tr_{L_1} = \{Raa, Rab, Rbb\}$$

Here R is the only symbol of its category. It can only be replaced by itself, which means that it can in effect be disregarded. So in this and similar examples to follow, when writing things like $X \subseteq Symb_{L_1}$, we really mean $X \subseteq Symb_{L_1} - \{R\}$.

L_1 has the feature that no replacement of a single symbol can turn a true sentence into a false one; only a permutation of a and b can do that. We have, for example,

(13) $\emptyset \Rightarrow_\emptyset Raa$, $\emptyset \Rightarrow_\emptyset Rbb$, $\emptyset \not\Rightarrow_\emptyset Rab$, but if $a \in X$ or $b \in X$, then $\emptyset \Rightarrow_X Rab$

Note that the first claim already shows that $\Rightarrow_\emptyset \neq \Rightarrow^{min}$. As to the last claim of (13), since we are only allowed to replace symbols outside X, in this case at most one of a and b can be replaced, so the conclusion cannot be falsified. The following is a complete description of the Bolzano consequence relations in L_1:

FACT 8. *In the language L_1:*
(i) $\Rightarrow_\emptyset \ = \ cl_{L_1}(\{\langle\emptyset, Raa\rangle, \langle\emptyset, Rbb\rangle\})$
(ii) *If a or b belong to $X \subseteq Symb_{L_1}$, then*
$\Rightarrow_X \ = \ cl_{L_1}(\{\langle\emptyset, Raa\rangle, \langle\emptyset, Rbb\rangle, \langle\emptyset, Rab\rangle\})$

Proof. (i) By (13) and monotonicity, we need only consider inferences with the conclusion Rab. (I.e. for all $\Gamma \subseteq Sent_{L_1}$, $\Gamma \Rightarrow_\emptyset Raa$ follows from $\emptyset \Rightarrow_\emptyset Raa$ by monotonicity.) Suppose $\Gamma \Rightarrow_\emptyset Rab$. We can assume $Raa, Rbb \notin \Gamma$, by (13) and transitivity: if e.g. $Raa \in \Gamma$ then $\Gamma - \{Raa\} \Rightarrow_\emptyset Rab$. By reflexivity and monotonicity, we can also assume that $Rab \notin \Gamma$. But $Rba \not\Rightarrow_\emptyset Rab$ (permuting a and b makes the premise true and the conclusion false). So all valid inferences $\Gamma \Rightarrow_\emptyset \varphi$ in L_1 belong to the closure of the two listed in (i). The proof of (ii) is similar, using base monotonicity ($\Rightarrow_\emptyset \subseteq \Rightarrow_X$), and the last claim of (13). □

As a consequence, note that

(14) $\Rightarrow_\emptyset \subsetneq \Rightarrow_{\{b\}} = \Rightarrow_{\{a\}} = \Rightarrow_{\{a,b\}}$

3.4.2. *The language L_2*

L_2 just adds one symbol (of the same category) to L_1, but no new false sentences:

$$Symb_{L_2} = \{R, a, b, c\}$$
$$Sent_{L_2} = \{Rxy \colon x, y \in \{a, b, c\}\}$$
$$Tr_{L_2} = \{Raa, Rbb, Rcc, Rab, Rac, Rbc, Rca, Rcb\}$$

First, clearly,

(15) $\emptyset \Rightarrow_\emptyset Raa,\ \emptyset \Rightarrow_\emptyset Rbb,\ \emptyset \Rightarrow_\emptyset Rcc$, but if $x \neq y$, then $\emptyset \not\Rightarrow_\emptyset Rxy$

Next,

(16) If $x \neq y$ and $Rxy \notin \Gamma$, then $\Gamma \not\Rightarrow_\emptyset Rxy$.

For if ρ maps x to b, y to a, and the remaining symbol to c, it is a *permutation* of $Symb_{L_2}$, and then no sentence except Rxy is mapped to Rba, so all premises in Γ are true.

With respect to $\Rightarrow_{\{a\}}$, we have, in addition to the valid inferences with $\Rightarrow_\emptyset$,

(17) a. $\emptyset \Rightarrow_{\{a\}} Rab$ and $\emptyset \Rightarrow_{\{a\}} Rac$
b. $Rca \Rightarrow_{\{a\}} Rcb$ and $Rba \Rightarrow_{\{a\}} Rbc$

Next, if $Rba \notin \Gamma$, then Γ is true, and so cannot imply Rba. Also

(18) $\{Rba, Rbc, Rcb\} \not\Rightarrow_{\{a\}} Rca$ [map a to itself, permute b and c]

For $\Rightarrow_{\{a,c\}}$, the situation is quite simple, since the empty set now implies each sentence except Rba, and no set of premises not containing Rba implies Rba. Our findings can be summarized as follows:

FACT 9. *Let* $\Phi_0 = \{\langle\emptyset, Rxx\rangle \colon x \in Symb_{L_2}\}$, *and* $\Phi_1 = \Phi_0 \cup \{\langle\emptyset, Rab\rangle, \langle\emptyset, Rac\rangle\}$. *In the language* L_2*:*

(i) $\Rightarrow_\emptyset \ = \ cl_{L_2}(\Phi_0)$

(ii) $\Rightarrow_{\{a\}} \ = \ cl_{L_2}(\Phi_1 \cup \{\langle\{Rba\}, Rbc\rangle, \langle\{Rca\}, Rcb\rangle\})$

(iii) $\Rightarrow_{\{a,c\}} \ = \ cl_{L_2}(\{\langle\emptyset, Rxy\rangle \colon (x, y) \neq (b, a)\}) \ = \ \Rightarrow_{\{a,b,c\}}$

4. Minimality

L_1 and L_2 provide examples where different sets X, Y generate the same Bolzano consequence. One would expect sets that are *minimal* in this respect to be particularly well behaved. Perhaps the most obvious idea about minimality is the following.

DEFINITION 10. *X is* minimal *iff for all* $u \in X$, $\Rightarrow_{X-\{u\}} \subsetneq \Rightarrow_X$.

So X is minimal in the sense that if any one of its symbols is left out, a smaller consequence relation results. The other natural sense of 'minimal', as we noted, is minimality with respect to the sets generating the same consequence relation. In fact, it is easy to see that these two notions of minimality are equivalent.

$\emptyset$ is minimal in any language. In the languages L_1 and L_2 of Section 3.4, all singleton sets are minimal, since in each case, $\Rightarrow_{\{x\}}$ is distinct from $\Rightarrow_{\emptyset}$. Also, it is easy to see that $\{a, c\}$ is minimal in L_2, but $\{a, b\}$ in L_1, and $\{a, b, c\}$ in L_2, are not minimal.

Being minimal doesn't entail being the smallest set generating the same consequence relation; e.g. in L_1 there is a set of symbols X with distinct minimal subsets in $\{Y : \Rightarrow_Y = \Rightarrow_X\}$. However, in Westersthl (2011) it was shown that there always exists at least one subset which is minimal among these when only compact consequence relations are considered:[15]

THEOREM 11. *For every $X \subseteq Symb_L$, if $\Rightarrow_X$ is compact, then X has a subset which is minimal among those generating $\Rightarrow_X$.*

Thus, if we restrict attention to minimal subsets of *Symb*, no compact consequence relation of the form $\Rightarrow_X$ will be left out.

In this paper, we shall prove related and in a sense stronger results. First, we show that under some additional assumptions about the language L, there actually is a *smallest* subset generating $\Rightarrow_X$; moreover, this subset has a simple independent description (Corollary 26). Then we prove that the same result holds if we lift those restrictions (but not compactness), but use a slightly more general framework for Bolzano consequence (Corollary 39).

The requirement of compactness in Theorem 11, however, cannot be removed, as we now show.

FACT 12. *There is a language L and a set $X \subseteq Symb_L$ such that $\Rightarrow_X$ is not compact, and there is no minimal X' with $\Rightarrow_X = \Rightarrow_{X'}$.*

Proof. We use a language $L_{\mathbb{N}}$ for arithmetic with numerals and predicates for any finite or co-finite set of natural numbers, plus a quantifier for "there are infinitely many". The *symbols* in $Symb_{L_{\mathbb{N}}}$ are

[15] Actually, the proof in Westersthl (2011) was given for finitary Bolzano consequence relations, but it is easily adapted to compact relations. That paper also identified a stricter notion of minimality, called *strong minimality*, and proved some results about it. These results are subsumed under the treatment in the present paper. In particular, in the more general Bolzano style framework introduced in Section 7, minimality and strong minimality coincide.

thus constants c_n for every $n \in \mathbb{N}$, predicates P_A and $\neg P_A$ for every *finite* set A of numbers, and a predicate functor *Inf*, taking predicates to sentences. (So the non-empty categories here are 'numeral', '1-place predicate' and 'predicate functor'.) The *sentences* in $Sent_{L_\mathbb{N}}$ are of one of the forms $P_A c_n$, $\neg P_A c_n$, $Inf P_A$, and $Inf \neg P_A$. A sentence φ is in $Tr_{L_\mathbb{N}}$ iff φ is $P_A c_n$ and $n \in A$, or φ is $\neg P_A c_n$ and $n \notin A$, or φ is $Inf \neg P_A$. Note that $L_\mathbb{N}$ is countable.

Let $X = \{c_n\}_{n \in \mathbb{N}}$. In what follows, $(\neg)P_A$ stands for an arbitrary predicate P_A or $\neg P_A$. First:

(19) If there are infinitely many sentences of the form $(\neg)P_A c_i$ in Γ, then $\Gamma \Rightarrow_X Inf(\neg)P_A$.

This is because a replacement that makes all the sentences in Γ true then has to replace $(\neg)P_A$ by a predicate with an infinite extension (that is, a predicate of the form $\neg P_B$), since the c_i must not be replaced. But then, after such a replacement, the conclusion is also true. On the other hand,

(20) If $Inf(\neg)P_A \notin \Gamma$, and only finitely many sentences of the form $(\neg)P_A c_i$ are in Γ, then $\Gamma \not\Rightarrow_X Inf(\neg)P_A$.

To see this, let A' be the finite set of numbers i such that $(\neg)P_A c_i \in \Gamma$. Consider the replacement ρ which replaces $(\neg)P_A$ by $P_{A'}$ and all other predicates by $\neg P_\emptyset$. Since $Inf(\neg)P_A \notin \Gamma$, it follows that all sentences in $\Gamma[\rho]$ are true (note that $Inf \neg P_\emptyset$ and all sentences $\neg P_\emptyset c_j$ are true), but $\varphi[\rho]$, i.e. $Inf P_{A'}$, is false. And since ρ does not act on X, this shows that $\Gamma \not\Rightarrow_X \varphi$. Next, we observe

(21) If $(\neg)P_A c_i \notin \Gamma$, then $\Gamma \not\Rightarrow_X (\neg)P_A c_i$.

For consider the replacement ρ which replaces $(\neg)P_A$ by $\neg P_{\{i\}}$ and all other predicates by $\neg P_\emptyset$. ρ doesn't act on X, all sentences in $\Gamma[\rho]$ are true (even if $Inf(\neg)P_A \in \Gamma$, since $Inf \neg P_{\{i\}}$ is true), but $(\neg)P_A c_i[\rho]$, i.e. $\neg P_{\{i\}} c_i$, is false.

This allows us to conclude:

(22) $\Rightarrow_X$ is not compact.

Take $\Gamma = \{P_{\{0\}} c_n\}_{n \in \mathbb{N}}$ and $\varphi = Inf P_{\{0\}}$. Then $\Gamma \Rightarrow_X \varphi$ by (19), but, by (20), there is no finite subset Γ' of Γ such that $\Gamma' \Rightarrow_X \varphi$.

Now let $X^- \subseteq X$ be any set of symbols such that the number of constants c_i which are in X but not in X^- is finite, and let $X^{--} \subseteq X$ be any set of symbols such that the number of constants c_i in X but not in X^{--} is infinite. Then we have:

(23) $\Rightarrow_{X^-} = \Rightarrow_X$

For suppose $\Gamma \Rightarrow_X \varphi$. We must show $\Gamma \Rightarrow_{X^-} \varphi$. This is clear if $\varphi \in \Gamma$, so suppose $\varphi \notin \Gamma$. It then follows from (21) that φ cannot be of the form $(\neg)P_A c_i$. So we have $\varphi = \mathit{Inf}(\neg)P_A$ for some A, and then it follows from (20) that there must be infinitely many sentences of the form $(\neg)P_A c_i$ in Γ. Thus, there are infinitely many sentences $(\neg)P_A c_i$ in Γ such that c_i is in X^-. So it is still the case that for a replacement ρ to make all the sentences in Γ true, ρ has to replace $(\neg)P_A$ by a predicate $\neg P_B$ with an infinite extension. Finally,

(24) $\Rightarrow_{X^{--}} \neq \Rightarrow_X$

Take for Γ all sentences of the form $P_{\{0\}}c_i$ for c_i not in X^{--}, and $\mathit{Inf}P_{\{0\}}$ for φ. By (19), $\Gamma \Rightarrow_X \varphi$, but now $\Gamma \not\Rightarrow_{X^{--}} \varphi$. Consider a replacement ρ such that $\rho(c_i) = c_0$ for all c_i not in X^{--}, but nothing else is moved. All sentences in $\Gamma[\rho]$ are true, since $\Gamma[\rho]$ is the singleton $\{P_{\{0\}}c_0\}$, but $\varphi[\rho]$, i.e. $\mathit{Inf}P_{\{0\}}$, is false.

Now the desired claim follows: there is no minimal subset X' of X such that $\Rightarrow_X = \Rightarrow_{X'}$. Subsets of X are either of the form X^- or X^{--}. But subsets of the form X^- are clearly not minimal, and subsets of the form X^{--} do not generate a consequence relation identical to $\Rightarrow_X$. $\Box$

5. Extracting constants from consequence relations

5.1. Defining extraction

We now introduce an operation corresponding to the extraction of logical constants from a consequence relation. When a particular consequence relation is given, certain symbols are to be considered as logical constants because the consequence relation makes them play a special role with respect to validity. As explained in the Introduction, our guiding intuition is that a symbol is constant if replacing it can destroy *at least one* inference.

DEFINITION 13. *The function* $C_{\_} : \mathit{CONS}_L \to \wp(\mathit{Symb}_L)$ *is defined for* $\Rightarrow \in \mathit{CONS}_L$ *by* $u \in C_{\Rightarrow}$ *iff there are* Γ, φ, *and* u' *such that* $\Gamma \Rightarrow \varphi$ *but* $\Gamma[u/u'] \not\Rightarrow \varphi[u/u']$.

We first observe, as a direct consequence of the Replacement Lemma, that when $C_{\_}$ is applied to a Bolzano consequence relation, it will never pick out a non-logical constant:

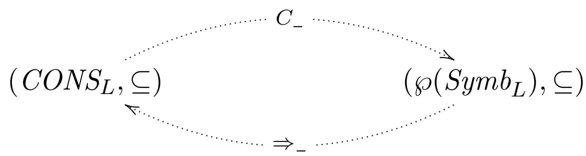

Figure 1. Logical consequence and constant extraction

FACT 14. *For all* $X \subseteq Symb$, $C_{\Rightarrow_X} \subseteq X$.

Proof. Suppose $u \in C_{\Rightarrow_X}$, and Γ, φ, and u' are as above. If $u \notin X$ we would have $\Gamma[u/u'] \Rightarrow_X \varphi[u/u']$ by Replacement. So $u \in X$. □

As discussed in Section 1, logical consequence can be construed as a function from sets of symbols to consequence relations. Extraction goes in the opposite direction. Moreover, the domains of both functions are naturally ordered by inclusion, so the situation is as shown in Figure 1. Fact 4(b) said that $\Rightarrow\_$ is an order-preserving mapping from $(\wp(Symb_L), \subseteq)$ to $(CONS_L, \subseteq)$. We would like $C\_$ to provide some sort of inverse order-preserving mapping. Before looking into this and other properties of $C\_$, let us see some examples of how $C\_$ works.

5.2. Examples

There is one case when the function $C\_$ trivially fails to yield the intended result because of its substitutional character, namely, when a symbol u is unique in its category. Then there is no other symbol to replace u with, so it will not count as a logical constant, no matter what inferential role it plays. This situation arises with negation, which is usually the only unary connective in logical languages. To sidestep this difficulty, we shall assume, when considering propositional logic or first-order logic, that they come equipped with another unary connective, say †, interpreted by the constant unary truth-function 'equal to false'.[16] With this assumption, we can verify that $C\_$ satisfies the first criterion mentioned in the Introduction for a reasonable 'extraction function': it gives the correct set of logical constants in familiar logical languages.

[16] As Lloyd Humberstone pointed out to us, we could have avoided this somewhat artificial manoeuvre if a more permissive notion of replacement had been used (cf. note 6).

5.2.1. *Familiar logical languages*
Indeed, things go smoothly for propositional and first-order logic.

FACT 15. *$C_{\models_{PL}}$ is the standard set of logical constants of propositional logic.*

Proof. $p \models_{PL} p \vee q$ but $p \not\models_{PL} p \wedge q$. That is, replacing $\vee$ by $\wedge$ destroys the validity of the first inference, so $\vee \in C_{\models_{PL}}$. Likewise, $p \models_{PL} \neg\neg p$ but $p \not\models_{PL} \dagger\dagger p$, and thus $\neg \in C_{\models_{PL}}$. Similarly for other familiar constants. On the other hand, (uniformly) replacing propositional letters can never destroy a valid $\models_{PL}$-inference. □

Recall from Section 3.2 that if the interpreted propositional language is non-trivial (the sequence of truth values of $p_0, p_1, \ldots$ is not eventually constant), then $\models_{PL}$ is a Bolzano consequence, say, $\models_{PL} = \Rightarrow_{\{\neg,\dagger,\vee,\wedge\}}$. But the fact that $C_{\_}$ recovers the right constants doesn't depend on this. We get, with the same kind of argument as above, the correct result also for first-order logic, even though $\models_{FO}$ is not usually a Bolzano consequence relation (see Section 3.3):

FACT 16. *$C_{\models_{FO}}$ is the standard set of logical constants of first-order logic.*

It should be clear why all standard logical constants will be extracted. Any occurrence of a name or a predicate symbol in a first-order validity is schematic, so no such symbol will be extracted. This is all there is to prove, since, following the convention adopted in Section 3.3, neither variables nor parentheses are considered as symbols that are to be tested for constancy. Such a convention is justified in so far as variables are viewed as syntactic markers encoding quantifier scope rather than as interpreted symbols, for which the question of constancy would make sense. This view gets some support by the fact that a variable-free notation for first-order logic may be used, as in Quine (1976).

More generally, most familiar consequence relations are such that suitably replacing a logical symbol can destroy an inference, while this is not possible for non-logical symbols. For example, consider an intuitionistic propositional logic whose consequence relation is defined not model-theoretically but axiomatically (with suitable extra axioms for †). Again, virtually the same kind of arguments show that $C_{\_}$ extracts precisely the logical symbols from this relation.

5.2.2. *Application to L_1 and L_2*
$C_\_$ behaves rather badly for L_1, since $C_{\Rightarrow_X} = \emptyset$ for all $X \subseteq Symb_{L_1}$. This is because replacing just *one* symbol can never destroy a $\Rightarrow_X$-inference in L_1; you need to replace two symbols simultaneously. Can we revise the extraction method so that it handles such cases better? There is actually a way (see Section 9.3), but in in this paper we shall stick to the function $C_\_$ and make it behave better by placing requirements on the language. For now, we observe that already in L_2, which contains just one extra symbol (of the same category), the situation is significantly different.

First, note that replacing a by c destroys the inference $\Rightarrow_{\{a\}} Rab$.[17] Thus, $a \in C_{\Rightarrow_{\{a\}}}$, so by Fact 14,

(25) $\quad C_{\Rightarrow_{\{a\}}} = \{a\}$

Next, since $\Rightarrow_{\{a,c\}} Rbc$ but $\not\Rightarrow_{\{a,c\}} Rba$, $c \in C_{\Rightarrow_{\{a,c\}}}$ in L_2. But $a \notin C_{\Rightarrow_{\{a,c\}}}$; this follows by checking that replacing a by b or c does not destroy any of the basic inferences listed in Fact 9 (iii). Thus, in L_2,

(26) $\quad C_{\Rightarrow_{\{a,c\}}} = \{c\}$

So the situation for L_2 is better than for L_1, but it is still not good, at least if we want $C_\_$ to be an order-preserving inverse on $CONS_L$. The failure of order preservation is no surprise given that there are both a positive and a negative condition in the definition of $C_\_$. The witness to a non-valid inference might disappear by shifting to a bigger consequence relation. Perhaps more surprisingly, the situation is no better for Bolzano consequences.

FACT 17. *There are languages L and sets $X, Y \subseteq Symb_L$ such that:*

(a) $\Rightarrow_X \subseteq \Rightarrow_Y$ *but* $C_{\Rightarrow_X} \not\subseteq C_{\Rightarrow_Y}$

(b) $\Rightarrow_X \not\subseteq \Rightarrow_{C_{\Rightarrow_X}}$

Proof. An example is provided by (25) and (26) for L_2. There we have $\Rightarrow_{\{a\}} \subseteq \Rightarrow_{\{a,c\}}$ by base monotonicity, but $\{a\} = C_{\Rightarrow_{\{a\}}} \not\subseteq C_{\Rightarrow_{\{a,c\}}} = \{c\}$. Also, $\Rightarrow_{\{a,c\}} \not\subseteq \Rightarrow_{C_{\Rightarrow_{\{a,c\}}}} = \Rightarrow_{\{c\}}$, since, for example, $\emptyset \Rightarrow_{\{a,c\}} Rab$ but $\emptyset \not\Rightarrow_{\{c\}} Rab$. □

[17] Here and in what follows we write $\Rightarrow_X \varphi$ rather than $\emptyset \Rightarrow_X \varphi$; meaning that φ is *valid* (relative to $\Rightarrow_X$).

6. A Galois connection under special assumptions

Fact 17 shows that $C\_$ does not yet behave in the way we would like. On the other hand, $C\_$ passes the first part of the test: it gives the right results when applied to familiar logical systems. Our diagnosis will be that the problems are due to particular features of the languages used in the counter-examples, rather than to shortcomings of the definition itself. The present section isolates a subclass of languages, sets of constants, and consequence relations for which $C\_$ behaves well, within the classical Bolzano framework introduced in Section 2. In the next section we will see that with a slight extension of that framework, many (but not all) of those restrictions can be lifted.

6.1. A factorization property for replacements

Let us take a closer look at the failure of monotonicity with respect to Bolzano consequence relations. In L_2, as we saw, $\Rightarrow_{\{a\}} \subseteq \Rightarrow_{\{a,c\}}$ but $C_{\Rightarrow_{\{a\}}} \nsubseteq C_{\Rightarrow_{\{a,c\}}}$, the reason being that $a \in C_{\Rightarrow_{\{a\}}}$ but $a \notin C_{\Rightarrow_{\{a,c\}}}$.

Relative to $\Rightarrow_{\{a\}}$ (and to $\Rightarrow_{\{a,c\}}$ as well), a *should* clearly be identified as a constant. After all, holding a fixed does make a difference, e.g. $\Rightarrow_{\{a\}} Rab$ but $\not\Rightarrow_{\emptyset} Rab$ (recall that Rba is false). But this is not sufficient for $C\_$ to spot a as a constant. $\Rightarrow_{\{a\}} Rab$ and $\not\Rightarrow_{\{a\}} Rba$, but Rab cannot be turned into Rba by replacing only a, as the definition of $C\_$ requires. For $\Rightarrow_{\{a\}}$, this is not a problem, because the non-constant symbol c can be used as a stop-over on the journey. Instead of jumping from the validity of Rab to the falsity of Rba, one can stop by the invalidity of Rcb. Then $a \in C_{\Rightarrow_{\{a\}}}$, because $\Rightarrow_{\{a\}} Rab$ and $\not\Rightarrow_{\{a\}} Rab[a/c]$.

Shifting to $\Rightarrow_{\{a,c\}}$, things are different: $\Rightarrow_{\{a,c\}} Rab[a/c]$. As it happens, there is no alternative way in L_2 to witness the constancy of a, and a ends up being outside $C_{\Rightarrow_{\{a,c\}}}$. But consider a language L_3 which is just as L_2 except that it contains another symbol d of the same category as c. So $Tr_{L_2} = Tr_{L_3} \cap Sent_{L_2}$, and let us also assume that Rad is true. In L_3, the situation improves because d can be used as a substitute stop-over: now $a \in C_{\Rightarrow_{\{a,c\}}}$, because $\Rightarrow_{\{a,c\}} Rab$ and $\not\Rightarrow_{\{a,c\}} Rab[a/d]$.

The lesson we would like to draw is that monotonicity holds when the language is rich enough (so that a d is available) and fails when it is not. We shall now consider a general factorization property for replacements which makes precise what is needed of such rich languages and which will enable us to prove monotonicity and more. The underlying idea is that, given two sets of symbols X and Y, performing a replacement outside X should be analyzable as first performing a replacement on the symbols outside X that are in Y and then performing a replacement on the remaining symbols outside Y. The fact that this can be done means that stop-overs are available.[18]

DEFINITION 18 (Factorization Property).

*Let $X, Y \subseteq Symb_L$ and $\Delta \subseteq Sent_L$. We say that X-*replacements in Δ factor through Y *iff for any replacement ρ which is defined on V_Δ and acts outside X, there are replacements σ and τ such that:*

(i) σ *acts only on* $Y - X$
(ii) $\sigma(Y - X) \cap V_\Delta = \emptyset$
(iii) τ *acts outside* Y
(iv) $\rho = \tau \circ \sigma$

Sentences Δ to be considered will typically be the sentences in an inference $\Gamma \Rightarrow_X \varphi$. $\sigma(Y - X) \cap V_\Delta = \emptyset$ then means that σ replaces symbols in $Y - X$ by 'new' symbols not occurring in the inference. In our earlier example, with $X = \{a\}$ and $Y = \{a, c\}$, σ corresponds to replacing c with d, so that everything that a replacement ρ which moves c could do can now be done by a replacement τ moving d instead of c.

When is this factorization possible, i.e. when are helpful symbols like d available? First, d qualified as a substitute for c because it was of the same category as c and did not belong to the old set of constants. In order to secure availability of such ds, a simple requirement would be that there are infinitely many symbols in each non-empty category. In Bonnay and Westerståhl (2010) we called such languages *rich*. But we also need that these rich resources cannot be all consumed by the chosen constants. For each non-empty category, there should always be infinitely many symbols in that category not taken as constants. The simplest requirement would be to restrict attention to *finite* sets of symbols. But we can replace these two by the single weaker requirement that we only consider *co-infinite* sets of symbols, i.e. in each non-empty category, there are infinitely many symbols in $Symb_L - X$. Let $\wp^{\text{coinf}}(Symb_L)$ be the set of such sets of symbols. As long as (in each non-empty category) $\wp^{\text{coinf}}(Symb_L)$ is not empty, assuming that the sets of symbols discussed are co-infinite entails assuming that L is rich.

In addition we need an assumption on the consequence relations. For simplicity, we shall assume that they are finitary, i.e. that only finite sets of premises are considered — marked by writing (FIN) —

[18] We say that a replacement *acts outside a set* if it is the identity on every element in that set for which it is defined, and *acts only on a set* if every element for which it is not the identity is in that set.

but in fact our proofs work under the weaker hypothesis that they are *compact* (we shall indicate the required changes in the proofs).[19]

Co-infiniteness of the set of constants and finiteness of the set of sentences guarantee that new symbols are available, so the factorization property holds:

LEMMA 19. *If* $Y \in \wp^{\mathrm{coinf}}(Symb_L)$ *and* Δ *is a finite set of* L*-sentences, then for all* $X \subseteq Symb_L$*,* X*-replacements in* Δ *factor through* Y*.*

Proof. Since Y is co-infinite in each non-empty category, so is $Y - X$. Since moreover Δ is finite, for every symbol a_i in $(Y - X) \cap V_\Delta$, there is a different symbol b_i which is of the same category as a_i but does not belong to V_Δ or to Y. Define σ by

$$\sigma(x) = \begin{cases} b_i & \text{if } x = a_i \\ x & \text{otherwise} \end{cases}$$

Then define τ on the range of σ by

$$\tau(x) = \begin{cases} \rho(a_i) & \text{if } x = b_i \\ \rho(x) & \text{otherwise} \end{cases}$$

It is easy to check that $\rho = \tau \circ \sigma$ and all other conditions in Definition 18 are satisfied. □

6.2. Monotonicity and preservation

The factorization property ensures monotonicity of $C_{\_}$ with respect to Bolzano consequence relations. The proof hinges on the same kind of reasoning we went through in the example.

THEOREM 20. (Fin) *If* Y *is co-infinite, then* $\Rightarrow_X \subseteq \Rightarrow_Y$ *implies* $C_{\Rightarrow_X} \subseteq C_{\Rightarrow_Y}$*.*

Proof. Assume $\Rightarrow_X \subseteq \Rightarrow_Y$, where Y is co-infinite, and $u \in C_{\Rightarrow_X}$. We want to show that $u \in C_{\Rightarrow_Y}$. By definition of $C_{\_}$, there are Γ, φ and u'

[19] Rather than (Fin), we could instead use the actually weaker requirement that the set of symbols occuring in an inference $\Gamma \Rightarrow \varphi$ is finite. Interestingly, our proofs do not go through if the set of symbols occurring in $\Gamma \cup \{\varphi\}$ is assumed to be co-infinite (in each non-empty category). This asymmetry suggests that the finiteness requirement does not play the same role for sets of symbols as it does for sets of premises. This is one reason we choose to work with the more precise if less simple assumption of co-infinity, rather than with richness and finite sets of symbols.

in L such that $\Gamma \Rightarrow_X \varphi$ and $\Gamma[u/u'] \not\Rightarrow_X \varphi[u/u']$. By definition of $\Rightarrow_X$, there is a replacement ρ acting outside of X such that the sentences in $\Gamma[u/u'][\rho]$ are true but $\varphi[u/u'][\rho]$ is false.

The hypotheses of Lemma 19 apply with respect to X, Y, and $\Delta = \Gamma \cup \{\phi\} \cup \Gamma[u/u'] \cup \{\phi[u/u']\}$, because of (FIN).[20] So X-replacements in Δ factor through Y, i.e. there are σ and τ such that σ acts only on $Y - X$, $\sigma(Y - X) \cap (V_\Delta) = \emptyset$, τ acts outside Y, and $\rho = \tau \circ \sigma$.

Since σ acts outside X we get, by Replacement,

$$\Gamma[\sigma] \Rightarrow_X \varphi[\sigma]$$

Hence, by assumption,

$$\Gamma[\sigma] \Rightarrow_Y \varphi[\sigma]$$

It is now sufficient to prove

(27) $$\Gamma[\sigma][u/\sigma(u')] \not\Rightarrow_Y \varphi[\sigma][u/\sigma(u')]$$

Since $\sigma(Y - X) \cap V_\Delta = \emptyset$, it follows that $[\sigma][u/\sigma(u')] = [u/u'][\sigma]$. Hence, since $\rho = \tau \circ \sigma$, $\Gamma[\sigma][u/\sigma(u')][\tau]$ is $\Gamma[\rho]$, a set of true sentences, and $\varphi[\sigma][u/\sigma(u')][\tau]$ is $\varphi[\rho]$, a false sentence. Since τ acts outside Y, this proves (27). □

In a similar manner, we can establish a more satisfactory inverse relationship between the mappings $\Rightarrow_\_$ and $C_\_$ restricted to Bolzano consequences. The consequence relation generated by *any* co-infinite set X of symbols is the same as what you get by first extracting the constants from $\Rightarrow_X$ and then generating the Bolzano consequence from those constants, even if they form a proper subset of X.

THEOREM 21. (FIN) *If X is co-infinite,* $\Rightarrow_X = \Rightarrow_{C_{\Rightarrow_X}}$.

Proof. $C_{\Rightarrow_X} \subseteq X$ already implies $\Rightarrow_{C_{\Rightarrow_X}} \subseteq \Rightarrow_X$, so all we need to prove is $\Rightarrow_X \subseteq \Rightarrow_{C_{\Rightarrow_X}}$. Assume $\Gamma \Rightarrow_X \varphi$. We must show $\Gamma \Rightarrow_{C_{\Rightarrow_X}} \varphi$. Let ρ be a replacement acting outside of $C_{\Rightarrow_X}$. It is sufficient to show

(28) $$\Gamma[\rho] \Rightarrow_X \phi[\rho]$$

The hypotheses in Lemma 19 apply, since X is co-infinite and $\Delta = \Gamma \cup \{\varphi\}$ is finite. So $C_{\Rightarrow_X}$-replacements in $\Gamma \cup \{\phi\}$ factor through X.

[20] If Γ were infinite but $\Rightarrow_X$ compact, the remainder of the proof would go through working with a finite subset Γ' of Γ such that $\Gamma' \Rightarrow_X \varphi$ and $\Gamma'[u/u'] \not\Rightarrow_X \varphi[u/u']$. Similarly for the proof of Theorem 21 below.

Hence we have σ and τ such that σ acts only on $X - C_{\Rightarrow_X}$, $\sigma(X - C_{\Rightarrow_X}) \cap V_\Delta = \emptyset$, τ acts outside X, and $\rho = \tau \circ \sigma$.

First, we show that

(29) $\Gamma[\sigma] \Rightarrow_X \varphi[\sigma]$

Since σ can be taken to be defined on the finite vocabulary of $\Gamma \cup \{\varphi\}$, σ acts on $X - C_{\Rightarrow_X}$, and no symbol in that set is replaced by a symbol occurring in $\Gamma \cup \{\varphi\}$, we have $\sigma = \sigma_n$ for some n, where, for $i \leq n$,

$$\sigma_i = id \cup \{(a_1, \sigma(a_1)), \ldots, (a_i, \sigma(a_i))\}$$

for some $a_1, \ldots, a_n \in X - C_{\Rightarrow_X}$, and id is the identity function on the rest of $V_{\Gamma\cup\{\varphi\}}$. Moreover, by the choice of the $\sigma(a_i)$, replacing $a_1, \ldots, a_n$ simultaneously according to σ and successively replacing them one by one gives the same result: for $\psi \in \Gamma \cup \{\varphi\}$,

$$\psi[\sigma_{i+1}] = \psi[\sigma_i][a_{i+1}/\sigma(a_{i+1})]$$

Assume for contradiction that a_{i+1} is the first symbol in the sequence for which consequence is not preserved, that is $\Gamma[\sigma_{i+1}] \not\Rightarrow_X \varphi[\sigma_{i+1}]$, but $\Gamma[\sigma_i] \Rightarrow_X \varphi[\sigma_i]$. So $\Gamma[\sigma_i][a_{i+1}/\sigma(a_{i+1})] \not\Rightarrow_X \varphi[\sigma_i][a_{i+1}/\sigma(a_{i+1})]$, but then $a_{i+1} \in C_{\Rightarrow_X}$, a contradiction. This proves (29).

Second, by Replacement, since τ acts outside X,

$$\Gamma[\sigma][\tau] \Rightarrow_X \varphi[\sigma][\tau]$$

Since $\rho = \tau \circ \sigma$, this proves (28). □

Theorem 21 relies on two assumptions: that X is co-infinite and that we consider only finitary (or compact) consequence relations. That none of these assumptions can be dropped follows from the next two facts.

FACT 22. (FIN) *There is a language L and an infinite set $X \subseteq Symb_L$ with finite complement such that $\Rightarrow_X \not\subseteq \Rightarrow_{C_{\Rightarrow_X}}$.*

Proof. Consider the language L_2', which is a rich variant of L_2: $Symb_{L_2'} = \{R, a, b, c_0, c_1, \ldots\}$, $Sent_{L_2'} = \{Rxy : x, y \in Symb_{L_2'}\}$, and $Tr_{L_2'} = Sent_{L_2'} - \{Rba\}$. Now let

$$X = \{a, c_0, c_1, ...\}$$

Then we claim

$$a \notin C_{\Rightarrow_X}$$

Otherwise, there would be a finite set Γ, and φ, u' such that $\Gamma \Rightarrow_X \varphi$ but $\Gamma[a/u'] \not\Rightarrow_X \varphi[a/u']$. The latter means that there would be a replacement of b only — since b is the only symbol outside X — such that $\Gamma[a/u'][b/b']$ is true and $\varphi[a/u'][b/b']$ is false. Now observe that $\varphi[a/u']$ does not contain a; otherwise $u' = a$ which contradicts the assumptions. If $b' = b$, then $\varphi[a/u'][b/b']$ does not contain a either. If $b' \neq b$, then $\varphi[a/u'][b/b']$ does not contain b. So $\varphi[a/u'][b/b']$ is a sentence of the form Rxy which does not contain both a and b. But all those sentences are true: contradiction.

Next, note that $\Rightarrow_X Rc_ia$ but $\not\Rightarrow_X Rba$. This shows that each c_i belongs to $C_{\Rightarrow_X}$, and so

(30) $\quad C_{\Rightarrow_X} = X - \{a\} = \{c_0, c_1, \ldots\}$

But now observe that $\Rightarrow_X Rab$ but $\not\Rightarrow_{X-\{a\}} Rab$. Together with (30) this proves $\Rightarrow_X \not\subseteq \Rightarrow_{C_{\Rightarrow_X}}$. □

FACT 23. *There is a language L and a co-infinite set* $X \subseteq Symb_L$ *such that* $\Rightarrow_X$ *is not compact and* $\Rightarrow_X \not\subseteq \Rightarrow_{C_{\Rightarrow_X}}$.

Proof. We use a variant $L'_{\mathbb{N}}$ of the arithmetical toy language $L_{\mathbb{N}}$ from the proof of Lemma 12 in Section 4. To the symbols of $L_{\mathbb{N}}$ we add predicate functors Inf_n for $n \geq 1$, letting $Inf_0 = Inf$. We also add new numerals z_n for $n \geq 0$. The predicate symbols are the same, and the intuitive idea is that each Inf_n means the same as Inf (i.e. 'is infinite'), c_i denotes the number i as before, and z_n may denote any number. The sentences have the same forms as in $L_{\mathbb{N}}$, using also the new predicate functors and numerals. Thus, for each finite $A \subseteq \mathbb{N}$, $Inf_n\neg P_A$ is true, Inf_nP_A is false, and if d is a numeral denoting i, then P_Ad is true iff $i \in A$, and $\neg P_Ad$ is true iff $i \notin A$.

As before, let $X = \{c_0, c_1, \ldots\}$. Note that in $L'_{\mathbb{N}}$, X is co-infinite (in its category). Now claims corresponding to (19) – (24) in the proof of Lemma 12 go through in $L'_{\mathbb{N}}$ as well. First, with the same proof, we have

(31) If there are infinitely many sentences of the form $(\neg)P_Ac_i$ in Γ, then $\Gamma \Rightarrow_X Inf_n(\neg)P_A$.

Likewise, with very small changes, we obtain

(32) If Γ contains no sentence of the form $Inf_m(\neg)P_A$, and only finitely many sentences of the form $(\neg)P_Ac_i$, then $\Gamma \not\Rightarrow_X Inf_n(\neg)P_A$.

Also, for any numeral d and finite set A,

(33) If $(\neg)P_A d \notin \Gamma$, then $\Gamma \not\Rightarrow_X (\neg)P_A d$.

For suppose d denotes the number i, and consider a replacement ρ mapping $(\neg)P_A$ to $\neg P_{\{i\}}$ and all other predicate symbols to $\neg P_\emptyset$, while Inf_n is mapped to Inf, d is mapped to c_i (so if d *is* c_i, it is not moved), all the other z_k are mapped to some c_j with $j \neq i$, and no c_k is moved. Since $(\neg)P_A d \notin \Gamma$, one readily checks that $\Gamma[\rho]$ is true but $\varphi[\rho] = \neg P_{\{i\}} c_i$ is false.

Now, we claim:

(34) $C_{\Rightarrow_X} = \emptyset$

It suffices to show that $c_i \notin C_{\Rightarrow_X}$, for all i. Otherwise, there are i, Γ, φ, and d such that $\Gamma \Rightarrow_X \varphi$ but $\Gamma[c_i/d] \not\Rightarrow_X \varphi[c_i/d]$. It follows that $\varphi \notin \Gamma$, and thus by (33), that φ is not of the form $(\neg)P_A d'$. So φ has to be $\mathit{Inf}_n(\neg)P_A$ for some A and n. Since $\mathit{Inf}_n(\neg)P_A$ is not in Γ, hence not in $\Gamma[c_i/d]$ either, no sentence of the form $\mathit{Inf}_m(\neg)P_A$ is in $\Gamma[c_i/d]$; this follows since obviously, for all m, n,

$$\mathit{Inf}_m(\neg)P_A \Rightarrow_X \mathit{Inf}_n(\neg)P_A$$

So no sentence of the form $\mathit{Inf}_m(\neg)P_A$ is in Γ, and then (32) implies that there are infinitely many sentences of the form $(\neg)P_A c_j$ in Γ, since $\Gamma \Rightarrow_X \varphi$. However, from (31) it follows that only finitely many sentences of this form belong to $\Gamma[c_i/d]$, since $\Gamma[c_i/d] \not\Rightarrow_X \varphi$. But this is impossible, since $\Gamma[c_i/d]$ results from Γ by replacing c_i in at most one such sentence. This proves (34).

The example $\Gamma = \{P_{\{0\}} c_n : n \in \mathbb{N}\}$ and $\varphi = \mathit{Inf}\, P_{\{0\}}$ shows as before that $\Rightarrow_X$ is not compact. It also shows that $\Rightarrow_X \not\subseteq \Rightarrow_{C_{\Rightarrow_X}} = \Rightarrow_\emptyset$, since $\Gamma \Rightarrow_X \varphi$, but, clearly, $\Gamma \not\Rightarrow_\emptyset \varphi$. □

6.3. A Galois connection

Let us take stock. What kind of correspondence do we get between $C_\_$ and $\Rightarrow_\_$? We wanted something as close as possible to an isomorphism, with as few assumptions as possible. A relevant notion of correspondence in that context is the notion of a *Galois connection.* A Galois connection is a quadruple $\langle \mathcal{A}, \mathcal{B}, f, g \rangle$ with $\mathcal{A}$ and $\mathcal{B}$ two ordered structures, $f : A \to B$ and $g : B \to A$ two functions, such that the following conditions hold:[21]

[21] A more compact characterization is that for all $a \in A$ and $b \in B$,

$$a \leq g(b) \text{ iff } f(a) \leq b$$

(I) f is monotone

(II) g is monotone

(III) $g \circ f$ is increasing

(IV) $f \circ g$ is decreasing

Even though Galois connections do not constitute full-blown isomorphisms, they 'contain' one: from (I)–(IV) one can prove that f is an isomorphism with inverse g between the well-behaved subsets $g(B)$ and $f(A)$ of A and B.

For an arbitrary language L, let $\mathcal{A} = (CONS_L, \subseteq)$, the set of all truth-preserving consequence relations in L ordered by inclusion, and $\mathcal{B} = (\wp(Symb_L), \subseteq)$, the set of all possible sets of constants ordered by inclusion. $C_\_$ and $\Rightarrow_\_$ are candidates for providing a Galois connection between $\mathcal{A}$ and $\mathcal{B}$. Base monotonicity (Fact 4) says that $\Rightarrow_\_$ is monotone — this is condition (II). The fact that the set of constants extracted from a Bolzano consequence relation $\Rightarrow_X$ is included in the original set X of constants (Fact 14) says that $C_{\Rightarrow_\_}$ is decreasing — this is condition (IV). Conditions (I) and (III) do not hold in general, not even (Fact 17) when attention is restricted from $CONS_L$ to the proper subset $BCONS_L$ of Bolzano consequence relations. However, for $BCONS_L$, suitable assumptions give us what we need: Theorem 20 is condition (I), and Theorem 21 implies that $\Rightarrow_{C_\_}$ is increasing, this is condition (III). Thus:

THEOREM 24. (FIN) *$C_\_$ and $\Rightarrow_\_$ constitute a Galois connection between $BCONS_L^{\text{coinf}}, \subseteq)$ and $(\wp^{\text{coinf}}(Symb_L), \subseteq)$.*

Here $BCONS_L^{\text{coinf}}$ is the set of consequence relations of the form $\Rightarrow_X$ for some $X \in \wp^{\text{coinf}}(Symb_L)$.

Our Galois connection is rather special in that the image of $\wp^{\text{coinf}}(Symb_L)$ under $\Rightarrow_\_$ is the whole of $BCONS_L^{\text{coinf}}$.[22] This reflects the fact that all of $CONS$ or $BCONS$ could not be part of the connection: restriction to the image of $\wp^{\text{coinf}}(Symb_L)$ under $\Rightarrow_\_$ is needed not only to get an isomorphism but already to satisfy conditions (I) and (III). Indeed, we do not have a characterization of the action of $C_\_$ on consequence relations which are not of the form $\Rightarrow_X$ for some X (but see the informal discussion in Section 9.2).

This is equivalent to the combination of (I)–(IV).

[22] This also corresponds to the fact that not only is $g \circ f$ increasing, but actually $g \circ f = Id_A$ as stated in Theorem 21.

Of special interest is now the image of $BCONS_L^{\mathrm{coinf}}$ under $C\_$. Which well-behaved subset of $\wp^{\mathrm{coinf}}(Symb_L)$ gets selected by the Galois connection to be the codomain of the isomorphism? The answer is given by the next result.

COROLLARY 25. (FIN) *The image under $C\_$ of $BCONS_L^{\mathrm{coinf}}$ is the set of minimal sets in $\wp^{\mathrm{coinf}}(Symb_L)$.*

Proof. First, to prove that every minimal co-infinite set X is the image of some $\Rightarrow_Y$ under $C\_$, we prove that it is the image of the consequence relation generated by itself, that is: If X is minimal and co-infinite, $X = C_{\Rightarrow_X}$. Because of Fact 14, we need only show $X \subseteq C_{\Rightarrow_X}$. This follows from $\Rightarrow_X = \Rightarrow_{C_{\Rightarrow_X}}$ (Theorem 21), since X is minimal and co-infinite.

Second, we prove: For every co-infinite X, $C_{\Rightarrow_X}$ is minimal. Take $u \in C_{\Rightarrow_X}$. We must show that $\Rightarrow_{C_{\Rightarrow_X}} \not\subseteq \Rightarrow_{C_{\Rightarrow_X}-\{u\}}$. By definition, there are Γ, φ, and u' such that $\Gamma \Rightarrow_X \varphi$ but $\Gamma[u/u'] \not\Rightarrow_X \varphi[u/u']$. So, by Theorem 21, $\Gamma \Rightarrow_{C_{\Rightarrow_X}} \varphi$. Also, by Replacement, $\Gamma \not\Rightarrow_{X-\{u\}} \varphi$. Since $C_{\Rightarrow_X} - \{u\} \subseteq X - \{u\}$ we get, by base monotonicity, $\Gamma \not\Rightarrow_{C_{\Rightarrow_X}-\{u\}} \varphi$. □

Thus, by general facts about Galois connections:

COROLLARY 26. (FIN) *$C\_$ is an isomorphism, with inverse $\Rightarrow\_$, from $(BCONS_L^{\mathrm{coinf}}, \subseteq)$ onto $(\wp^{\mathrm{coinf}}(Symb_L), \subseteq)$ restricted to minimal sets.*

Assuming (FIN) but without the restriction to co-infinite sets of symbols, there is for every X at least one minimal set generating the same consequence relation as X (Theorem 11), but uniqueness is not guaranteed. With the supplementary assumption that only co-infinite sets are considered, Corollary 26 says that $C_{\Rightarrow_X}$ is the unique minimal set generating the same consequence relation as X.

All the results in this section are made possible by considering only special languages (the rich ones), special consequence relations (the finite or compact ones) and special sets of constants (the co-infinite ones). Rather than making specific assumptions such as these, another way to get results would be to work with a more general definition of $\Rightarrow\_$ that would encapsulate what is necessary to get Lemma 19. This alternative route is explored in the next section.

7. Languages permitting expansions

Richness, or co-infinity, is all about having available symbols *in the language* L. But these symbols are just, as we said, 'stop-over' symbols enabling us to spot logical constants; they play no other role in L. It may seem *ad hoc*, or even unreasonable, to require of an interpreted language that it contain such an unlimited supply of extra symbols. It would be much more reasonable to have a mechanism for *adding* them whenever needed. Instead of 'staying in L', one would work with suitable *expansions* of L. We shall slightly revise our Bolzano set-up to make this possible.

This is also a further step towards a Tarskian model-theoretic framework. In such a framework, merely expanding the language is always *conservative* in the sense that the consequence relation for the old language is not affected. As is clear from the previous sections, this may fail drastically in a substitutional setting.[23] We now eliminate this obvious limitation of the classical substitutional framework, while still remaining in a Bolzano style setting.[24]

There might be the following worry: If we start with an interpreted language, and then expand this language in various ways, how do we know what the new symbols (and sentences) *mean*? In principle, the answer is: we are free to *stipulate* what they mean, as long as this doesn't 'disturb' the meanings of symbols (and sentences) in L. In fact, we shall see that for the applications in this section, each new symbol we introduce can be taken to be *synonymous* with some L-symbol, in the precise sense that interchanging occurrences of these two symbols never changes the truth values of sentences containing them. Such expansions will be called *expansions with copies*. Then, the extra feature added to the Bolzano framework is merely to allow free introduction of new names for old things.

7.1. Expansions

Recall that, for each language L, $Cat_L = \{C_L : C \in Cat\}$ partitions $Symb_L$.

[23] For example, expand the language L_1 by adding a new symbol c such that Rac is false. Then, although $\Rightarrow_{\{a\}} Rab$ holds in L_1, it fails in the expanded language.

[24] The idea is not new; it was proposed, for example, in Bonevac (1985), in the context of first-order logic. There the motivation was to be able to talk about uncountable domains in a countable language with substitutional interpretation of the quantifiers. Most of Bonevac's paper is about arguing that it is natural to consider expansions in a substitutional setting; we can only agree.

DEFINITION 27. *We say that L' is an* expansion *of L, in symbols $L \leq L'$, iff*

$Symb_L \subseteq Symb_{L'}$

For each category $C \in Cat$, $C_L \subseteq C_{L'}$

$Sent_L = \{\varphi \in Sent_{L'} : V_\varphi \subseteq Symb_L\}$

$Tr_L = Tr_{L'} \cap Sent_L$

One easily verifies that

(35) $\leq$ is a partial order (reflexive, antisymmetric, and transitive).

A partially ordered set Z is *directed* iff it is upward closed: if $a, b \in Z$ there is $c \in Z$ such that $a \leq c$ and $b \leq c$. Now, our idea is to start as before with a fixed language L, but also consider a directed family $\mathcal{L}$ of expansions of L. We must then reformulate what have done so far accordingly. First, here is the new notion of Bolzano consequence:

DEFINITION 28. For $\Gamma \cup \{\varphi\} \subseteq Sent_L$ and $X \subseteq Symb_L$: $\Gamma \Rrightarrow_{X,L} \varphi$ iff for every $L' \in \mathcal{L}$ and every replacement ρ in L' (for Γ and φ) which is the identity on X, if $\Gamma[\rho] \subseteq Tr_{L'}$, then $\varphi[\rho] \in Tr_{L'}$.

The family $\mathcal{L}$ is suppressed in this notation, and has to be made clear in context. If $\mathcal{L} = \{L\}$, we have our previous notion of Bolzano consequence: $\Rrightarrow_{X,L} = \Rrightarrow_X$. But in general, $\Rrightarrow_{X,L} \subsetneq \Rrightarrow_X$.

Normally, the sentences we talk about will belong to several languages in $\mathcal{L}$. But since consequence is defined in terms of all expansions (in $\mathcal{L}$) of a given language, this is not a problem. That is, we now have the conservativity property for expansions that fails in the old setting (cf. note 23):

LEMMA 29. (Conservativity Lemma) *If $\Gamma \cup \{\varphi\} \subseteq Sent_L$, $X \subseteq Symb_L$, and $L' \in \mathcal{L}$, then*

$$\Gamma \Rrightarrow_{X,L} \varphi \;\; \textit{iff} \;\; \Gamma \Rrightarrow_{X,L'} \varphi$$

or, equivalently,

$$\Rrightarrow_{X,L} \; = \; \Rrightarrow_{X,L'} \restriction Sent_L$$

where the right-hand side is relative to the subfamily $\mathcal{L}' = \{L'' \in \mathcal{L} : L' \leq L''\}$.

Proof. If there is a counter-example, via a replacement in some $L'' \geq L'$, to $\Gamma \Rrightarrow_{X,L'} \varphi$, then there is one to $\Gamma \Rrightarrow_{X,L} \varphi$ as well, since $L \leq L''$.

Conversely, if there is a counter-example, via a replacement in some $L''' \geq L$, to $\Gamma \Rrightarrow_{X,L} \varphi$, choose, by directedness, L'''' such that $L''' \leq L''''$ and $L' \leq L''''$. Then we have a counter-example in L'''' (with the same replacement) to $\Gamma \Rrightarrow_{X,L'} \varphi$. □

In what follows, when $\mathcal{L}$ is given and $L' \in \mathcal{L}$, we always understand $\Rrightarrow_{X,L'}$ to be relative to the corresponding subfamily generated by L'. Note that each $\Rrightarrow_{X,L'}$ belongs to $CONS_{L'}$, but also, by the Conservativity Lemma, to $CONS_{L''}$ for each expansion L'' of L'. Moreover, relations of this form are base monotone, and (straightforwardly adjusted versions of) the Replacement and Occurrence lemmas hold.

7.2. Useful classes of expansions

Our revised notions of consequence, extraction, etc. (see below) work for any directed class $\mathcal{L}$ of expansions of L. In particular, let

$exp(L)$

be the class of *all* expansions of L, and let

$copies(L)$

be the class of *expansions with copies* of L, i.e. expansions such that each new symbol is *synonymous*, in the sense indicated above, to some L-symbol. It is straightforward to verify that $(copies(L), \leq)$ is also a directed partial order.

Our toy language L_2 is an expansion of L_1, but not an expansion with copies. To make c a copy of b, both of Rba and Rca must be false, not just Rba as in L_2.

We say that a class $\mathcal{L}$ of expansions of L is *full*, if for all sets $\{a_i : i \in I\} \subseteq Symb_L$ there is an expansion $L' \in \mathcal{L}$ and distinct symbols $b_i \in Symb_{L'} - Symb_L$ of the same category as a_i, for $i \in I$. Clearly, $copies(L)$ (and hence $exp(L)$) is full.

Suppose $\Gamma \Rrightarrow_{X,L} \varphi$, L' is an expansion with copies of L, and Γ', φ' result from Γ, φ by replacing some occurrences of L-symbols with copies in L'. We *cannot* conclude that $\Gamma' \Rrightarrow_{X,L'} \varphi'$, for it may be the case that some but not all occurrences of an L-symbol u have been replaced by a copy $\overline{u}$ (or distinct occurrences by distinct copies), and a replacement of u and $\overline{u}$ by distinct symbols may then yield a counter-example that was not available before. One easily verifies, however, that the following converse holds:

(36) With $L, L', \Gamma, \Gamma', \varphi, \varphi'$ as above: if $\Gamma' \Rrightarrow_{X,L'} \varphi'$, then $\Gamma \Rrightarrow_{X,L} \varphi$.

7.3. General consequence relations

Definition 28 associates with each $X \subseteq Symb_L$ not just one consequence relation, but a *conservative family* of consequence relations, one for each $L' \in \mathcal{L}$. Such families can be seen as instances of a new notion of consequence. As before, $\mathcal{L}$ is a directed family of expansions of a base language L.

DEFINITION 30. A *general consequence relation* (for $\mathcal{L}$) is a family of consequence relations (in the old sense) $\Rrightarrow = \{\Rightarrow^{L'}\}_{L' \in \mathcal{L}}$ such that for all $L', L'' \in \mathcal{L}$ with $L' \leq L''$, $\Rightarrow^{L'} = \Rightarrow^{L''} \restriction Sent_{L'}$.

General consequence relations are partially ordered under a generalized notion of inclusion: define

(37) $\quad \Rrightarrow \sqsubseteq \Rrightarrow'$ iff for all $L' \in \mathcal{L}$, $\Rightarrow^{L'} \subseteq \Rightarrow'^{L'}$

Furthermore, we write, when $\Gamma \cup \{\varphi\} \subseteq Sent_L$, $\Gamma \Rrightarrow \varphi$ instead of $\Gamma \Rightarrow^{L} \varphi$. By conservativity, this is equivalent to $\Gamma \Rightarrow^{L'} \varphi$ holding for all $L' \in \mathcal{L}$ (or for some $L' \in \mathcal{L}$).[25]

General Bolzano consequence relations are of course prime examples of general consequence relations, and we shall write $\Rrightarrow_X$ for the family $\{\Rrightarrow_{X,L'}\}_{L' \in \mathcal{L}}$ of consequence relations generated from $X \subseteq Symb_L$ and $\mathcal{L}$ according to Definition 28. General consequence relations of this form satisfy base monotonicity, and the Replacement and Occurrence Lemmas hold.

Next, the notion of minimality is as before: $X \subseteq Symb_L$ is *minimal* iff for each $u \in X$, $\Rrightarrow_X \not\sqsubseteq \Rrightarrow_{X-\{u\}}$. Again it is clear that minimality coincides with being minimal among the sets generating the same general consequence relation.

[25] The notation is handy, but strictly speaking it means that we are using '$\Rrightarrow$' in two senses: as a family of consequence relations and as a consequence relation. Thus, we employ $\sqsubseteq$ for the partial order among such families, but

$$\Rrightarrow \subseteq \Rrightarrow'$$

is used as before for the inclusion relation between (ordinary) consequence relations, meaning that if $\Gamma \Rrightarrow \varphi$ then $\Gamma \Rrightarrow' \varphi$; a weaker claim than $\Rrightarrow \sqsubseteq \Rrightarrow'$. Likewise, let us agree to use

$$\Rrightarrow = \Rrightarrow'$$

for equality between the consequence relations (i.e. $\Rightarrow^{L} = \Rightarrow'^{L}$), and instead

$$\Rrightarrow \equiv \Rrightarrow'$$

for equality between the families (i.e. for all $L' \in \mathcal{L}$, $\Rightarrow^{L'} = \Rightarrow'^{L'}$).

We say that a general consequence relation $\Rrightarrow = \{\Rightarrow^{L'}\}_{L' \in \mathcal{L}}$ is *compact* if each $\Rightarrow^{L'}$ is compact. The proof of Theorem 11 in (Westersthl, 2011) is easily modified to give:

THEOREM 31. *For every $X \subseteq Symb_L$, if the general consequence relation $\Rrightarrow_X$ is compact, then X has a subset which is minimal among those generating $\Rrightarrow_X$.*

We shall, however, obtain another proof of this theorem in the next subsection.

Finally, we generalize the definition of $C_{\_}$ to general consequence relations of the form $\Rrightarrow = \{\Rightarrow^{L'}\}_{L' \in \mathcal{L}}$. Let $u \in Symb_L$.

DEFINITION 32. *$u \in C_{\Rrightarrow}$ iff for some $L' \in \mathcal{L}$, $u \in C_{\Rightarrow^{L'}}$.*

Thus, $C_{\Rrightarrow}$ may properly include $C_{\Rightarrow_L}$, since the inference that gets destroyed by replacing u may belong to a proper expansion of L.

We now have the two 'easy' Galois conditions:

(38) a. If $X \subseteq Y$, then $\Rrightarrow_X \sqsubseteq \Rrightarrow_Y$ [base monotonicity]

b. $C_{\Rrightarrow_X} \subseteq X$ [by Replacement as before]

7.4. The Galois connection liberated

With expansions available, we don't have to worry about sufficiently many symbols being in the base language L. More precisely, Lemma 19 now holds without the restriction to co-infinite sets of symbols or finite sets of sentences.

In the remainder of this section, let $\mathcal{L}$ be any full directed class of expansions of L (Section 7.2).

LEMMA 33. *If Δ is any set of L-sentences, then for all $X, Y \subseteq Symb_L$, there is an expansion $L' \in \mathcal{L}$ such that in L', X-replacements in Δ factor through Y.*

Proof. Let $(Y - X) \cap V_\Delta = \{a_i : i \in I\}$. Since $\mathcal{L}$ is full, some expansion $L' \in \mathcal{L}$ contains for each a_i a distinct symbol b_i outside L of the same category. The rest of the proof is exactly as the proof of Lemma 19. □

As a result, we obtain monotonicity of $C_{\_}$ (Theorem 20) without the previous restrictions.

THEOREM 34. $\Rrightarrow_X \sqsubseteq \Rrightarrow_Y$ *implies* $C_{\Rrightarrow_X} \subseteq C_{\Rrightarrow_Y}$.

Proof. The proof is essentially the same, but we repeat it to indicate the use of expansions. Thus, assume $\Rrightarrow_X \sqsubseteq \Rrightarrow_Y$ and $u \in C_{\Rrightarrow_X}$. We must show that $u \in C_{\Rrightarrow_Y}$. By definition, there is an expansion L' of L in $\mathcal{L}$, and $\Gamma \cup \{\varphi\} \subseteq Sent_{L'}$ and $u' \in Symb_{L'}$ such that $\Gamma \Rrightarrow_{X,L'} \varphi$ but $\Gamma[u/u'] \not\Rrightarrow_{X,L'} \varphi[u/u']$. Thus, there are $L'' \geq L'$ in $\mathcal{L}$ and a replacement ρ in L'' acting outside of X, such that $\Gamma[u/u'][\rho] \subseteq Tr_{L''}$ but $\varphi[u/u'][\rho] \notin Tr_{L''}$.

By Lemma 33 with respect to X, Y and $\Delta = \Gamma \cup \{\phi\} \cup \Gamma[u/u'] \cup \{\phi[u/u']\}$, there is $L''' \geq L''$ in $\mathcal{L}$ such that in L''', X-replacements in Δ factor through Y. So there are σ and τ such that σ acts only on $Y - X$, $\sigma(Y - X) \cap (V_\Delta) = \emptyset$, τ acts outside Y, and $\rho = \tau \circ \sigma$. By Replacement and conservativity,

$$\Gamma[\sigma] \Rrightarrow_{X,L'''} \varphi[\sigma]$$

and thus, by hypothesis,

$$\Gamma[\sigma] \Rrightarrow_{Y,L'''} \varphi[\sigma] \tag{39}$$

Then we can show exactly as before that

$$\Gamma[\sigma][u/\sigma(u')] \not\Rrightarrow_{Y,L'''} \varphi[\sigma][u/\sigma(u')] \tag{40}$$

(39) and (40) entail that $u \in C_{\Rrightarrow_Y}$. □

Similarly, by following the earlier proof, inserting expansions at suitable points, we get a new version of Theorem 21. However, at one crucial step in that proof (the proof of (29)), it is required that $V_{\Gamma \cup \{\varphi\}}$ is finite (not just that it is co-infinite). Therefore, we still need the assumption (Fin), or at least compactness, for this result:

THEOREM 35. (Fin) *For every* $X \subseteq Symb_L$, $\Rrightarrow_X \equiv \Rrightarrow_{C_{\Rrightarrow_X}}$.

Interestingly, although compactness plays no role for the monotonicity of $C\_$ in the expansions framework, it cannot be dropped in Theorem 35:

FACT 36. *There is a language L and a full directed class of expansions of L with respect to which, for some* $X \subseteq Symb_L$, $\Rrightarrow_X$ *is not compact, and* $\Rrightarrow_X \not\sqsubseteq \Rrightarrow_{C_{\Rrightarrow_X}}$.

Proof. Consider again the language $L_{\mathbb{N}}$ defined in the proof of Fact 12, and let $\mathcal{L} = copies(L_{\mathbb{N}})$. As before, take $X = \{c_0.c_1, \ldots\}$. Now suitable versions of (19) – (21) will hold, in fact with proofs very similar to those for (31) – (33) in the proof of Fact 23; note that the language $L'_{\mathbb{N}}$ considered there was an expansion with copies of $L_{\mathbb{N}}$. We give some indications. First,

(41) If L is an expansion with copies of $L_{\mathbb{N}}$, $\Gamma \subseteq Sent_L$, and infinitely many sentences of the form Qc_i belong to Γ (where Q is a 1-place predicate symbol in L), then $\Gamma \Rrightarrow_{X,L} Inf\,Q$.

The proof is as before, except that a replacement ρ such that all sentences in $\Gamma[\rho]$ are true may now replace Q by a new predicate symbol $\rho(Q)$. But since $\rho(Q)$ must be a copy some $(\neg)P_A$, and since $\rho(Inf)$ must be a copy of *Inf*, it follows in the same way that $\rho(Inf)\rho(Q)$ is true. Next,

(42) If L is an expansion with copies of $L_{\mathbb{N}}$, $Inf\ Q \notin \Gamma$, and if only finitely many sentences of the form Qc_i belong to Γ, then $\Gamma \not\Rrightarrow_{X,L} Inf\,Q$.

(43) If L is an expansion with copies of $L_{\mathbb{N}}$ and $Qd \notin \Gamma$, then $\Gamma \not\Rrightarrow_{X,L} Qd$.

The proof of (43) uses essentially the replacement used for (33) in the proof of Fact 23. Now we can follow the argument for (34) in that proof to establish

(44) $C_{\Rrightarrow_X} = \emptyset$

Then we have (from (41) with $L = L_{\mathbb{N}}$) that $\{P_{\{0\}}c_n : n \in \mathbb{N}\} \Rrightarrow_X Inf\,P_{\{0\}}$. This inference gives a counter-example to compactness as before. But non-compactness also follows from Theorem 35, together with the obvious fact that $\{P_{\{0\}}c_n : n \in \mathbb{N}\} \not\Rrightarrow_{\emptyset} Inf\,P_{\{0\}}$, which establishes that $\Rrightarrow_X \not\sqsubseteq \Rrightarrow_{C_{\Rrightarrow_X}}$. □

The corollaries of Theorems 34 and 35 follow just as in Section 6.3. Let $GBCONS_L$ be the set of general consequence relations of the form $\Rrightarrow_X$ for some $X \subseteq Symb_L$.

THEOREM 37. (FIN) *$C_\_$ and $\Rrightarrow_\_$ constitute a Galois connection between $(GBCONS_L, \sqsubseteq)$ and $(\wp(Symb_L), \subseteq)$.*

COROLLARY 38. (FIN) *The image under $C_\_$ of $GBCONS_L$ is the set of minimal sets in $\wp(Symb_L)$.*

COROLLARY 39. (FIN) *$C_\_$ is an isomorphism with inverse $\Rrightarrow_\_$ from $(GBCONS_L, \sqsubseteq)$ onto $(\wp(Symb_L), \subseteq)$ restricted to minimal sets.*

Thus, as before, under (FIN) (or compactness) we have shown that $C_{\Rrightarrow_X}$ is the unique minimal set generating the same consequence relation as X. In particular, we have a (new) proof of Theorem 31.

8. From Bolzano to Tarski

Finally, we extend our results in the previous section to cover the more familiar Tarskian semantic notion of logical consequence, as given in (Tarski, 1936). This hinges on the fact that substitutional consequence is equivalent to semantic consequence when quantification over expansions is allowed.

8.1. Tarskian interpreted languages

Up to now, our interpreted languages L came equipped only with a set of true sentences. No more was needed to define a substitutional notion of consequence 'à la Bolzano'. For a Tarski style notion of consequence we also need a notion of interpretation for a language and a notion of truth with respect to interpretations. Accordingly, we now introduce *Tarskian interpreted languages*, which come equipped with interpretations, and we assume that a general definition of truth with respect to an interpretation is available for the family of languages under consideration. We shall assume as little as possible regarding the nature of interpretations and the truth relation.

For each (syntactic) category C, let S_C be a corresponding semantic category, intended to be the class of possible semantic values for symbols of category C.

DEFINITION 40. *A* Tarskian interpreted language *is a triple* $L = \langle Symb_L, Sent_L, I_L \rangle$*, where* $Symb_L$ *and* $Sent_L$ *are as before, and* I_L *is an* L-interpretation*, i.e. a function mapping each symbol* $u \in Symb_L$ *of category* C *to a semantic value* $I(u)$ *in* S_C*.* I_L *is called the* standard interpretation *of* L*.*

Let $\mathcal{I}_L$ be the class of L-interpretations. We assume that the general truth definition yields, for each Tarskian interpreted language L, a truth relation $\models_L \subseteq \mathcal{I}_L \times Sent_L$. Defining

$$Tr_L = \{\varphi \in Sent_L : I_L \models_L \varphi\}$$

we see that Tarskian interpreted languages are special cases of interpreted languages: the case when every symbol has its standard interpretation. Note, however, that in contrast with the more familiar situation in model-theoretic semantics, an interpretation here interprets *all* symbols of the language, not just the 'non-logical' ones.

When I and I' are two interpretations and X is a set of symbols,[26]

$$I =_X I'$$

means that I and I' agree on symbols in X, that is for all $u \in X$, $I(u) = I'(u)$. Our only requirement on the truth definition is that truth should be *local* in the following sense:

(45) If ρ is a replacement such that for all $u \in V_\varphi$, $I(u) = I'(\rho(u))$, then $I \models \varphi$ iff $I' \models \varphi[\rho]$.

Locality means that the question whether a sentence is true or not depends only on the semantic values of its symbols. In particular, if $I =_{V_\varphi} I'$, then $I \models \varphi$ iff $I' \models \varphi$. Arguably, any reasonable truth definition makes truth local — if semantic values do not determine truth values, they are not semantic values. Thus, we do not consider any other component in interpretations over and above semantic values. For example, there is in the present set-up no varying domain of interpretation which could make the truth value of sentences vary even when the semantic values of their symbols remain the same.[27]

8.2. The semantic notion of logical consequence

Tarski's semantic definition of logical consequence as preservation of truth under all possible reinterpretations of non-logical constants can be stated for a Tarskian interpreted language L in the usual way:

DEFINITION 41.
φ is a logical consequence of Γ with respect to a set of symbols X,

$$\Gamma \models_{X,L} \varphi\,,$$

iff for all interpretations J such that $J =_X I_L$, if $J \models \Gamma$, then $J \models \varphi$.

Given a Tarskian interpreted language L, substitutional consequence $\Rightarrow_X$ and semantic consequence $\models_{X,L}$ may be compared. As we already

[26] We drop L as a prefix or subscript when no ambiguity arises.

[27] We take this to be consonant with the original definition of logical consequence in Tarski (1936), which, in contrast to the modern model-theoretic one, was also given for an interpreted language and did not mention varying domains. Actually, the question whether changes in a domain's size were considered by Tarski at the time is a matter of dispute among Tarski scholars; see, for example, Gómez-Torrente (1996). Independently of historical issues, and for the sake of generality, one could think of ways to encode domain variations in the changes of semantic values, but we shall not pursue that here.

recalled, the substitutional definition makes logical consequence depend on the availability of symbols in L. An inference might be valid just because some semantic values needed to provide a counter-example are not the interpretations of any L-symbols. By contrast, the semantic definition makes all semantic values available by allowing for arbitrary reinterpretation. Thus, $\Gamma \models_{X,L} \varphi$ implies $\Gamma \Rightarrow_X \varphi$, but the converse is not true in general. However, the substitutional definition acquires a semantic flavor when expansions come into play as they did in the previous section.

The Tarskian consequence relation $\models_{X,L}$ relative to a set X of constants was defined as usual: there is no need to mention expansions of the interpreted language L since its symbols can be *re*interpreted. But just like other interpreted languages, Tarskian interpreted languages *can* be expanded. The definition is the same as before, except that we require $I_L = I_{L'} \upharpoonright Symb_L$ instead of $Tr_L = Tr_{L'} \cap Sent_L$. Locality guarantees that the former implies the latter. As a consequence, Tarskian expansions are a special kind of expansions. Given a Tarskian language L, we can consider the family

$$exp_T(L)$$

of all its Tarskian expansions. One can easily check that it is a full and directed family. Now substitutional consequence with respect to $exp_T(L)$ becomes equivalent to semantic consequence:

LEMMA 42. *With respect to* $\mathcal{L} = exp_T(L)$, $\Gamma \models_{X,L} \varphi$ *iff* $\Gamma \Rightarrow_{X,L} \varphi$.

Proof. From left to right: assume $\Gamma \models_{X,L} \varphi$. Let L' be a Tarskian expansion of L and ρ a replacement in L' acting outside X. We need to show that if $I_{L'} \models \Gamma[\rho]$, then $I_{L'} \models \varphi[\rho]$. Define an L'-interpretation J by

$$J(u) = \begin{cases} I_{L'}(\rho(u)) & \text{if } u \in dom(\rho) \\ I_L(u) & \text{otherwise} \end{cases}$$

By definition of J and locality, for any $\psi \in Sent_L$, $J \models \psi$ iff $I_{L'} \models \psi[\rho]$. But $I_L =_X J$, therefore $\Gamma \models_{L,X} \varphi$ implies that if $J \models \Gamma$, then $J \models \varphi$. Hence if $I_{L'} \models \Gamma[\rho]$, then $I_{L'} \models \varphi[\rho]$, as required.

From right to left: assume $\Gamma \Rightarrow_{X,L} \varphi$. Let J be an interpretation such that $I_L =_X J$. We need to show that if $J \models \Gamma$ then $J \models \varphi$. We define a Tarskian expansion L' by adding a copy u' of each symbol u for which $J(u) \neq I_L(u)$. Copies have the same interpretation as the symbols they are copies of, that is, we set $I_{L'}(u') = J(u)$. Now consider the replacement ρ which maps each such u to u' and is the identity elsewhere. Again, by locality, for any $\psi \in Sent_L$, $J \models \psi$ iff $I_{L'} \models \psi[\rho]$. Therefore, since $I_{L'} \models \Gamma[\rho]$ implies $I_{L'} \models \varphi[\rho]$, $J \models \Gamma$ implies $J \models \varphi$, as required. □

It follows that the general consequence relation $\Rrightarrow_X = \{\Rrightarrow_{X,L'}\}_{L' \in exp_T(L)}$ can also be written $\models_X = \{\models_{X,L'}\}_{L' \in exp_T(L)}$. In this case there is essentially only a notational difference between the consequence relation $\models_{X,L}$ and the family of consequence relations it generates, since each $\models_{X,L'}$ is defined independently of the expansions of L'. Still, since $\models_X$ *can* be seen as a general consequence relation in our sense, and since $exp_T(L)$ is a full family, Theorem 37 applies. Let $TCONS_L$ be the set of general consequence relations of the form $\models_X$ for some $X \subseteq Symb_L$, where L is a Tarskian interpreted language.

THEOREM 43. (Fin) *$C\_$ and $\models\_$ constitute a Galois connection between $(TCONS_L, \sqsubseteq)$ and $(\wp(Symb_L), \subseteq)$.*

This happy ending stems from a double virtue of expansions. On the one hand, they allow the Galois connection to hold. On the other hand, they allow semantic consequence to be reduced to substitutional consequence. Even though the problem in both cases amounts to circumventing potential limitations in the richness of the language, expansions do not play exactly the same role in the two cases. To get the equivalence between semantic consequence on the one hand and the substitutional definition of consequence with quantification over expansions on the other, expansions have to be *semantically rich*, they need to provide enough symbols to make all semantic values available. To get the Galois connection, expansions have to be *syntactically rich*, they need to have enough new symbols for the purely syntactic factorization property (Lemma 33) to hold.

9. Further perspectives

9.1. Where we are

The extraction procedure hardwired in the definition of $C\_$ can rightly be taken to satisfy the two adequacy criteria mentioned in the Introduction. First, $C\_$ yields results in accordance with our intuitions when applied to standard examples of logical consequence relations. Second, extraction thus defined does provide an inverse to the process of generating a consequence relation from a set of constants. This claim was made mathematically precise by means of the concept of a Galois connection. In particular, if one allows expansions to play a role in the definition $\Rrightarrow$ of consequence, $C_{\_}$ turns out to constitute a Galois inverse to $\Rrightarrow_X$ (on compact consequence relations defined on a full family of expansions). This was eventually shown to cover the familiar case of compact Tarskian consequence relations. In these settings, the role of $C_{\_}$ on $\Rrightarrow_X$ is to pick out a unique minimal set of constants.

We shall end by noting some potential limitations of the definition of $C_{\_}$ and, reflecting on them, suggest a few leads for further work. Our extraction procedure can be claimed to be both quite liberal and quite severe. We say that u is constant if it occurs essentially in at least one valid inference, in the sense that one can get to an invalid inference by replacing that symbol and nothing else. The phrase 'occurs essentially in *at least one* valid inference' in the definitional clause is responsible for the liberality. Is one inference enough for constancy?[28] The phrase 'by replacing that symbol *and nothing else*' is responsible for the severity. Why should one replace only one thing at a time?

9.2. Analytic and logical consequence

The liberality in the definition of $C_{\_}$ shows up in that $C_{\Rightarrow}$ will usually declare many more inferences valid than $\Rightarrow$ does (at least for many relations $\Rightarrow$ not of the form $\Rightarrow_X$). The reason is that $\Rightarrow$ might include some meaning postulates for a symbol u, even though it does not treat u as a logical constant. The kind of scenario we have in mind is one where $\Rightarrow$ *partly* fixes the interpretation of a symbol u by declaring valid some inferences essentially involving u, but cannot be construed as being of the form $\Rightarrow_X$ for some X with $u \in X$. In such a scenario, u will belong to $C_{\Rightarrow}$, so that $\Rightarrow_{C_{\Rightarrow}}$, contrary to $\Rightarrow$, relies on keeping the denotation of u *completely* fixed.

As a consequence, $C_{\_}$ cannot be used to tell the difference between logical inferences and merely analytic inferences. One might have hoped that C would select *logical* constants, in a way such that the further application of $\Rightarrow_{\_}$ would have isolated a core of purely logical inferences. Thus, $\Rightarrow_{C_{\Rightarrow}}$ would have been a subset of $\Rightarrow$, the subset of its purely logical inferences, whereas inferences in $\Rightarrow - \Rightarrow_{C_{\Rightarrow}}$ would have been the analytic ones. But this is not what is happening. To the contrary, $\Rightarrow_{C_{\Rightarrow}}$ will go beyond $\Rightarrow$ by exploiting all the information that can be gained from the fixed interpretation of constants in $C_{\Rightarrow}$.

The upshot is that $C_{\_}$ may do a good job at spotting *constants* with respect to validity – those symbols that matter to the validity

[28] As we noted in Section 1.2, the stronger version (which would read 'in *all* valid inferences') is not easily workable because of the necessary qualification regarding valid inferences whose validity is not due to the purported constant.

or invalidity of inferences – but that it does not ground a distinction between two kinds of constants, namely the logical constants properly speaking as opposed to symbols that merely come up with some meaning postulates attached to them. A natural question is then whether it is possible to define in a similar vein an operation that would select among all the constants in $C_{\Rightarrow}$ all and only the logical ones. We shall leave this for future work.

9.3. Non-uniform consequence

Let us turn to $C_{\_}$ being too severe. In principle, it seems that nothing precludes the role played by a constant u to show up only in connection with other substitutions. In that case, $C_{\_}$ would fail to select u. By contrast, one might consider a different extraction procedure, say $C^*_{\_}$. As before, $u \in C^*_{\Rightarrow}$ would require finding a valid inference $\Gamma \Rightarrow \varphi$ in which u occurs. But now the invalid inference which is to witness u's essential involvement in that validity could be obtained by means of a replacement ρ which moves u (as before) but possibly other symbols as well. However, this cannot be the whole story. It could not yet capture the fact that u was essential to the validity of $\Gamma \Rightarrow \varphi$, since, after all, putting some other symbol in place of u could be totally contingent to the destruction of the inference. We need to require that substituting u was indeed necessary for ρ to do so. This leads to the following definition: $u \in C^*_{\Rightarrow}$ iff there is an inference $\Gamma \Rightarrow \varphi$ and a replacement ρ such that $\Gamma[\rho] \not\Rightarrow \varphi[\rho]$ but $\Gamma[\rho_{-u}] \Rightarrow \varphi[\rho_{-u}]$, where ρ_{-u} is the replacement which differs from ρ at most on u and maps u to itself.

$C^*_{\_}$ is indeed less severe than $C_{\_}$. It is easy to check that, for any $\Rightarrow$, $C_{\Rightarrow} \subseteq C^*_{\Rightarrow}$.[29] The converse is not true in general, as witnessed by our language L_1. We had $a \notin C_{\Rightarrow_{\{a\}}}$ but we get $a \in C^*_{\Rightarrow_{\{a\}}}$. To see this, recall that $\Rightarrow_{\{a\}} Rab$ (and $\Rightarrow_{\{a\}} Raa$) but $\not\Rightarrow_{\{a\}} Rba$. Let ρ swap a and b. We get $\Rightarrow_{\{a\}} Rab$, $\not\Rightarrow_{\{a\}} Rab[\rho]$ and $\Rightarrow_{\{a\}} Rab[\rho_{\{-a\}}]$. Not uninterestingly, this suggests that C^* solves some of the problems on account of which we had to introduce rich languages or full expansions. By way of ρ, no stop-over is needed, so that $a \in C^*_{\Rightarrow_{\{a\}}}$ not only in the context of L_2 (where c is available) but already in L_1 (where no symbol different from a and b is available).

We shall not engage here in a thorough examination of the properties of $C^*_{\_}$. Despite what was pointed out in the previous paragraph, $C^*_{\_}$ does not yield a straightforward Galois correspondence for $\Rightarrow$ or $\models_X$. However, let us mention that it can be shown that $C^*_{\_}$ does yield a

[29] As a consequence, $C^*_{\_}$ cannot help with the conceptual difficulties surrounding the difference between logical and analytic inferences.

straightforward Galois connection for another notion of logical consequence. The notion we have in mind is stronger than the standard one in that it allows for *non-uniform* replacements of non-logical constants. Accordingly, a classical tautology such as $p \vee \neg p$ ceases to be valid, since $p \vee \neg q$ is not valid. This stronger notion of logical consequence has recently received a lot of interest from linguists who are looking for a connection between logicality and grammaticality (not all validities or contradictions are ungrammatical, but validity or contradiction seems to play a role in some sentences being ungrammatical).[30] It is a rather pleasant surprise that this notion independently appears in connection with the extraction problem and the contrast between $C_{\_}$ and $C^*_{\_}$.

Acknowledgements

Both authors were supported by the ESF-funded project 'Logic for Interaction' (LINT; http://www.illc.uva.nl/lint/index.php), a Collaborative Research Project under the Eurocores program LogICCC, and by a grant from the Swedish Research Council.

References

Abrusán, M.: 2011, 'Presuppositional and negative islands: a semantic account'. *Natural Language Semantics* **19**(3), 257–321.

Aczel, P.: 1990, 'Replacement systems and the axiomatization of situation theory'. In: R. Cooper, K. Mukai, and J. Perry (eds.): *Situation Theory and its Applications, Vol 1.* Stanford: CLSI Publications, pp. 3–33.

Bar-Hillel, Y.: 1950, 'Bolzano's definition of analytic propositions'. *Theoria* **16:2**, 91–117.

Bolzano, B.: 1837, *Theory of Science.* Dordrecht, 1973: ed. Jan Berg, D. Reidel.

Bonevac, D.: 1985, 'Quantity and quantification'. *Noûs* **19**, 229–247.

Bonnay, D.: 2008, 'Logicality and invariance'. *Bulletin of Symbolic Logic* **14:1**, 29–68.

Bonnay, D. and D. Westerståhl: 2010, 'Logical consequence inside out'. In: M. Aloni and K. Schulz (eds.): *Amsterdam Colloquium 2009.* Heidelberg: LNAI 6042, Springer, pp. 193–202.

Carnap, R.: 1937, *The Logical Syntax of Language.* London: Kegan, Paul, Trench Trubner & Cie. Rev. ed. translation of *Logische Syntax der Sprache*, Wien, Springer, 1934.

Dunn, M. and N. Belnap: 1968, 'The substitution interpretation of the quantifiers'. *Noûs* **4**, 177–185.

[30] The idea was introduced by Gajewski under the name of *L-analyticity* in connection with 'there' sentences and exceptives (Gajewski, 2002). It has then been taken up to help explain various other phenomena, including measurement scales (Fox and Hackl, 2006) and presuppositional or negative islands (Abrusán, 2011).

Feferman, S.: 2010, 'Set-theoretical invariance criteria for logicality'. *Notre Dame Journal of Formal Logic* **51**, 3–20.

Fox, D. and M. Hackl: 2006, 'The universal density of measurement'. *Linguistics and Philosophy* **59:5**, 537–586.

Gajewski, J.: 2002, 'L-analyticity and natural language'. Manuscript.

Gentzen, G.: 1932, 'Über die Existenz unabhängiger Axiomensysteme zu unendlichen Satzsystemen'. *Mathematische Annalen* **107**, 329–350. English translation in *The Collected Papers of Gerhard Gentzen*, ed. M. E. Szabo, North-Holland, Amsterdam, 1969.

Gómez-Torrente, M.: 1996, 'Tarski on logical consequence'. *Notre Dame Journal of Formal Logic* **37:1**, 125–151.

Hertz, G.: 1923, 'Über Axiomensysteme für beliebige Satzsysteme'. *Mathematische Annalen* **87**, 246–269.

Lewis, C. I. and C. H. Langford: 1932, *Symbolig Logic.* New York: Dover.

MacFarlane, J.: 2009, 'Logical Constants'. In: E. N. Zalta (ed.): *The Stanford Encyclopedia of Philosophy.* The Metaphysics Research Lab Center for the Study of Language and Information Stanford University, fall 2009 edition.

Peters, S. and D. Westerståhl: 2006, *Quantifiers in Language and Logic.* Oxford: Oxford University Press.

Quine, W.: 1976, 'Algebraic Logic and Predicate Functors'. In: *The Ways of Paradox.* Cambridge, Mass: Harvard University Press, pp. 283–307.

Shoesmith, D. J. and T. J. Smiley: 1978, *Multiple-Conclusion Logic.* Cambridge: Cambridge University Press.

Tarski, A.: 1930a, 'Fundamentale Begriffe der Methodologie der deduktiven Wissenschaften. I'. *Monatshefte für Mathematik und Physik* **37**, 361–404. English translation in (**?**).

Tarski, A.: 1930b, 'Über einige fundamentale Begriffe der Metamathematik'. *Comptes rendus des séances de la Societeé des Sciences et des Lettres de Varsovie* **23**, 22–29. English translation in (**?**).

Tarski, A.: 1936, 'On the concept of logical consequence'. In: *Logic, Semantics, Metamathematics.* Clarendon Press, Oxford. Republished 1983 by Hackett Publishing, Indianapolis, pp. 409–420.

Tarski, A.: 1986, 'What are logical notions?'. *History and Philosophy of Logic* **7**, 145–154.

van Benthem, J.: 2003, 'Is there still logic in Bolzano's key?'. In: E. Morscher (ed.): *Bernard Bolzanos Leistungen in Logik, Mathematik und Physik Bd. 16.* Sankt Augustin: Academia Verlag, pp. 11–34.

Westersthl, D.: 2011, 'From constants to consequence, and back'. *Synthese.* (in press). DOI 10.1007/s11229-011-9902-z.