Classical vs. modern squares of opposition, and beyond*

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Abstract

The main difference between the classical Aristotelian square of opposition and the modern one is not, as many seem to think, that the classical square has or presupposes existential import. The difference lies in the relations holding along the sides of the square: (sub)contrariety and subalternation in the classical case, inner negation and dual in the modern case. This is why the modern square, but not the classical one, applies to any (generalized) quantifier of the right type: *all, no, more than three, all but five, most, at least two-thirds of the*, ... After stating these and other logical facts about quantified squares of opposition, we present a number of examples of such squares spanned by familiar quantifiers. Special attention is paid to possessive quantifiers, such *Mary's, at least two students'*, etc., whose behavior under negation is more complex and in fact can be captured in a *cube* of opposition.

Keywords: generalized quantifiers, monotonicity, possessive quantifiers, square of opposition

1 Background

1.1 Squares

A square of opposition illustrates geometrically a particular way that negation interacts with certain operators. In the *quantified square* (Figure 1), the operators are the Aristotelian quantifiers, but several other operators follow a similar pattern (Figure 2). In these squares – here in addition a *modal* square, a *tem-*

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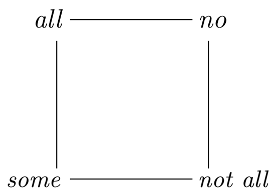


Figure 1: The quantified square

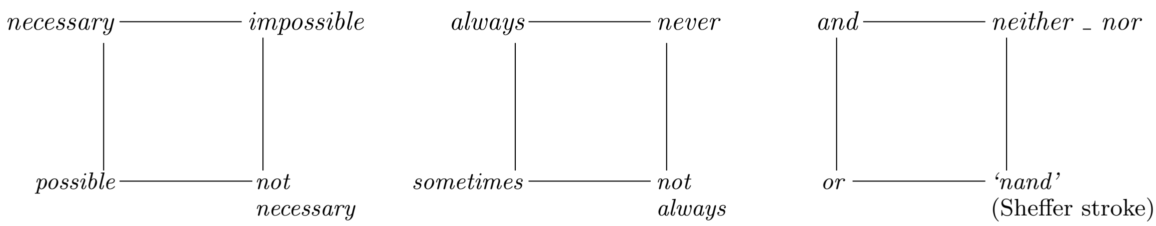


Figure 2: Various squares

poral square, and a *propositional* square – you have ordinary, *contradictory* or *outer* negation along the diagonals, and various weaker forms of negation as well as other logical relations along the edges of the square. Precisely what these other relations are is a matter of debate, as we will see presently. But an easily identifiable rough pattern is common to all these squares. For example, the *deontic* square replaces *necessary* by *obligatory*, *impossible* by *prohibited*, and *possible* by *permitted*.

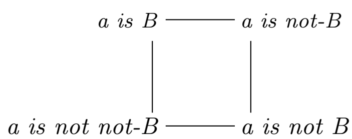
In a clear sense, the quantified square is the fundamental one: it recurs when you spell out the meaning of the operators in the other squares. For example, a conjunction is true iff *all* its conjuncts are true, and a disjunction is true iff *some* of the disjuncts are. Likewise, a proposition is necessary iff it is true under *all* circumstances (in *all* possible worlds), it is possible if true under *some* circumstance, and impossible if true under *none*.[1]

1.2 The Aristotelian square

In this paper we focus on the quantified square, and more generally on the interaction between negation and quantification. Let us draw the classical quantified square, that I will simply call the *Aristotelian square* (its content was described in words by Aristotle, and it was drawn as a square by Apuleios and Boethius some 800 years later), in somewhat more detail. In Figure 3, the only non-standard notation is all_{ei}, which is the universal quantifer with *existential import*, i.e. such that $all_{\mathrm{ei}}(A,B)$ is true iff A is non-empty and included in B. This is what Aristotle and most of his medieval followers, in agreement with many philosophers and most linguists today, took words like "all" and "every" to mean. In contrast, Frege and most logicians after him take them to stand for the quantifier *all* without existential import: $all(A,B) \Leftrightarrow A \subseteq B$. I come back to this issue presently, but let me for the record first describe the content of the Aristotelian square.

The (medieval) naming **A**, **E**, **O**, **I** of the four corners is handy and will be used consistently in what follows. As to the logical relations depicted in the square, we have, aside from contradictory negation along the two diagonals, the relations of contrariety, subcontrariety, and subalternation along the sides. These are seen as relations not between the quantifiers themselves but the corresponding statements made with them. Thus, φ and ψ are *contrary* iff

[1]One square whose contents were discussed by Aristotle falls partly outside this pattern. This is the *singular* square (Horn, 1989, §§1.1, 7.2–3):



The crucial notion here is that of *predicate negation*. E.g. one may distinguish *not-white* (or *not white*) as a 'logical' contrary of *white*, whereas *black* is a 'polar' contrary (along a color scale), and *brown* yet another kind of contrary (it is a contrary since nothing can be both white and brown). There are intricate linguistic (and possibly logical) issues involved, but they don't have much to do with quantification, and I shall disregard them here.

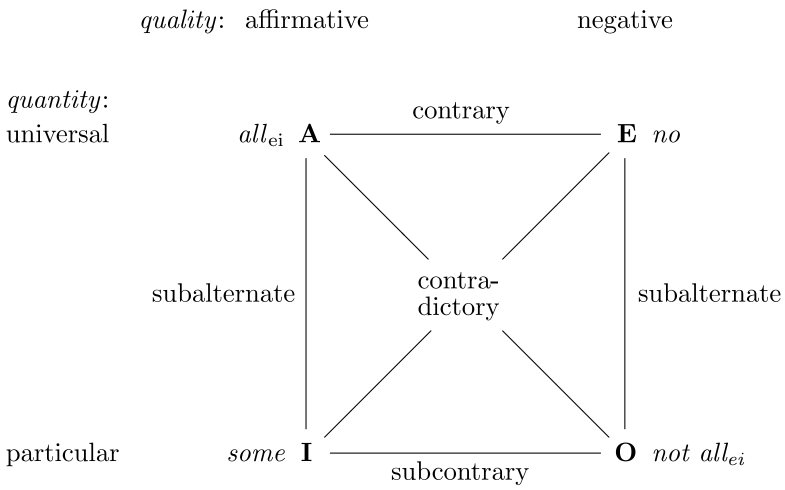


Figure 3: The Aristotelian square

they cannot both be true, *subcontrary* iff they cannot both be false, and ψ is *subalternate* to φ iff whenever φ is true, so is ψ (i.e. iff φ implies ψ).[2]

That the **A** and **E** corners are contrary clearly shows that the universal quantifier must be taken to have existential import in this square. The same conclusion follows from the fact that **A** is held to imply **I**. But whereas few linguists or philosophers have had doubts about this, there has been much uneasiness about the consequence that the quantifier at the **O** corner, the contradictory negation of all_{ei}, then must have the truth conditions: *not* $all_{\mathrm{ei}}(A, B)$ iff *either* A is empty *or* something in A is not in B. It doesn't seem very plausible that a sentence of the form "Not all As are B" would mean that. Various remedies have been suggested. One, that may initially appear to have some plausibility, is to say that *all* corners have existential import. Thus, for example, "No As are B" would amount to "There are As, but none of them are B". Alternatively, one might even stipulate that 'empty terms' are simply prohibited.

I will not follow any of these routes. My main reason is a simple (Gricean) distinction between the *truth-conditions* of a sentence and when it is *appropriate to utter* it. If I know that there are no girls in your class, it is inappropriate for me to say

(1) No girls in your class came to the party.

But it surely isn't *false*. Having been at the party, I may utter (1) appropriately, believing there are girls in your class; your informing me that there aren't any doesn't compel me to *retract* (1). Retracting it would amount to saying

[2]One often adds the requirement that φ and ψ *can* both be false in the definition of contrariety, and likewise that φ and ψ *can* both be true in the definition of subcontrariety. Everything I say below about these notions holds for the revised definitions as well.

(2) Some girls in your class came to the party.

but I know this is false. The distinction is nowadays accepted by most theorists, I think. For another example, if I know that Mary is at home, it is inappropriate for me to say

(3) Mary is either at home or at the movies.

It is even misleading to say so, but it clearly isn't false. No one concludes from the inappropriateness of (3) in that situation that A is sometimes true even though $A \vee B$ is false!

So we shouldn't take *no* to have existential import. (The case of *all* is slightly more complex and will be returned to below.[3]) As indicated in the diagram above, statements can be classified according to *quality*: they are either *affirmative* or *negative*; negative statements don't make existential claims in the Aristotelian square, but affirmative ones do. Another classification concerns *quantity*: in an obvious sense, a statement (of the relevant form) is either *universal* or *particular*.

1.3 Plan

A number of square-related issues have been discussed in the literature, some of them since the days of Aristotle. For example,

(a) What (if anything) is less valuable/real/natural/informative about negative statements as compared to positive ones?

(b) What (if anything) makes a statement negative rather than positive?

(c) Why is the **O** corner never lexically realized (in any language)?

(d) Can the corners be distinguished/ordered in terms of the simplicity (conceptual, computational, or other) of the corresponding quantifiers?

(e) Does the **A** corner have existential import?

These issues will not be discussed, other than indirectly, here (except that I make a brief remark on (d) in section 5).[4] Instead, I will focus on the modern variant

[3] Aristotle's own position on empty terms may be unclear, but many medieval philosophers had no inhibitions whatever against them. A nice example is Paul of Venice (c. 1400, quoted from Parsons (2004)), who claimed that the sentence

(i) Some man who is a donkey is not a donkey.

is *true*, since $A =$ the set of men who are donkeys is empty. This in turn is an argument that the truth conditions at the **O** corner are as we described them above, even if one uses the form "some _ not" rather than "not all".

[4] Horn (1989) is a classic history of negation, to which I refer for discussion of the above and many other issues related to negation. For recent contributions to the discussion, in terms of the idea of a natural logic for language, see Jaspers (2005) and Pieter Seuren's Natural Logic Project (a description is a available at his home page). While Horn (and many others)

of the Aristotelian square. But my main aim is not to argue in favor of the Fregean version of those four quantifiers. Rather, my point is that the modern square, in contrast with the Aristotelian one, represents forms of negation that are much more ubiquitous in language than the usual discussion reveals, and that this particular square is just one example among indefinitely many others with the same pattern of negation but with other quantifiers at the corners. Indeed, any (generalized) quantifier (of type $\langle 1, 1 \rangle$), and thus any interpretation of any simple or complex determiner, spans a modern square (but not a classical one).

So my first task is to show that the 'oppositions' introduced by Aristotle, i.e. (sub)contrariety and subalternation, though not uninteresting, are not quite the ones to focus on in the study of natural language negation. The distinction may seem subtle for Aristotle's square, but it becomes obvious as soon as you consider squares with other quantifiers. The second part of the paper, then, examplifies how these new squares are manifested in language, and revisits some of the old issues in this more general context. I end with a particularly interesting example, that of possessive determiners, whose interpretation involves two quantifiers. This allows a new form of negation, and is seen to be representatble in a *cube* of opposition rather than a square.

2 Classical vs. modern squares

The modern or Fregean version of the Aristotelian square is drawn in Figure 4. At first sight, one may get the impression that the only difference is that

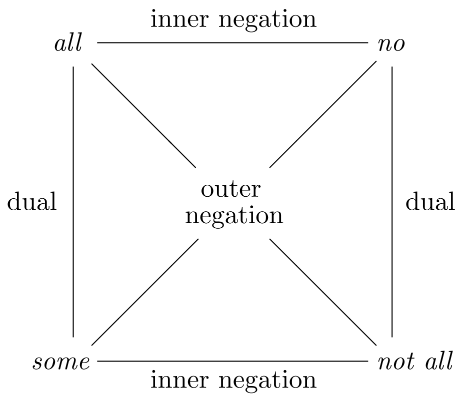


Figure 4: The modern version of Aristotle's square

considers (a) and (b) to be non-issues, he offers an interesting explanation of (c). Löbner (1990) and Jaspers (2005) lay great emphasis on (d) (though their respective answers differ). A pragmatic view on (e) is defended in Horn (1997), and, in a different way, in (Peters and Westerståhl, 2006, ch. 4.2.1). But, as we will see, there are more interesting issues of existential import for quantifiers other than those in Aristotle's square.

all has replaced $all_{\rm ei}$ at the **A** and **O** corners (I continue to use these names of the corners, but without writing them in the diagrams). Likewise, if one disregards empty terms, the two squares seem to coincide. So the difference might appear to boil down simply to different views about existental import. But this impression is *completely misleading*, and it stems from considering only the four Aristotelian quantifiers. Instead, the main difference consists in the different choice of relations along the sides (not the diagonal) of the square. Existential import is a side issue in this context.

In the modern square there are only two relations along the sides: inner negation and dual, and like outer negation they can be seen as operations on the quantifiers themselves:

(4) a. The *outer negation* of Q, $\neg Q$, is defined by: $\neg Q(A,B) \Leftrightarrow$ not $Q(A,B)$

b. The *inner negation* of Q, $Q\neg$, is defined by: $Q\neg(A,B) \Leftrightarrow Q(A, M - B)$ (where M is the universe)

c. The *dual* of Q, $Q^{\rm d}$, is defined by: $Q^{\rm d} = \neg(Q\neg) = (\neg Q)\neg$

One sees that these relations do indeed hold as indicated in Figure 4. So it is the same relation along both horizontal sides, in contrast with the Aristotelian square, but more importantly, this relation has little to do with (sub)contrariety. In general, as we will see, nothing prevents Q and $Q\neg$ from being both true, or both false, of the same arguments. Likewise, the relation along the vertical sides has little or nothing to do with subalternation, since we may easily have Q true and $Q^{\rm d}$ false of the same arguments. The fact that $all_{\rm ei}(A,B)$ implies $some(A,B)$, and hence $\neg no(A,B)$, is just incidental to that particular square.

To appreciate these points, we need a technical concept.

2.1 Type $\langle 1, 1\rangle$ quantifiers

I only give the bare definitions necessary for what follows; for (much) more about quantifiers, see Peters and Westerståhl (2006).

(5) **Definition.** A (generalized) *quantifier Q of type $\langle 1,1\rangle$* associates with each universe M a binary relation Q_M between subsets of M.

Many such quantifiers interpret simple or complex determiners, and we can name them by the corresponding determiner phrase in italics (in, say, English). Here are a few examples, starting with some already mentioned ($|X|$ is the cardinality of the set X). For all M and all $A, B \subseteq M$,

$$all_M(A,B) \iff A \subseteq B$$
$$(all_{\rm ei})_M(A,B) \iff \emptyset \neq A \subseteq B$$
$$no_M(A,B) \iff A \cap B = \emptyset$$
$$\textit{at least two}_M(A,B) \iff |A \cap B| \geq 2$$
$$\textit{exactly five}_M(A,B) \iff |A \cap B| = 5$$

$$all\ but\ three_M(A,B) \iff |A-B| = 3$$
$$more\ than\ two\text{-}thirds\ of\ the_M(A,B) \iff |A \cap B| > 2/3 \cdot |A|$$
$$most_M(A,B) \iff |A \cap B| > |A-B|$$
$$the\ ten_M(A,B) \iff |A| = 10\ \&\ A \subseteq B$$
$$John\text{'}s_M(A,B) \iff \emptyset \neq A \cap \{a : \text{John `possesses' } a\} \subseteq B$$
$$no\ \_\ except\ John_M(A,B) \iff A \cap B = \{\text{John}\}$$
$$some\ but\ not\ all_M(A,B) \iff A \cap B \neq \emptyset \neq A-B$$
$$infinitely\ many_M(A,B) \iff A \cap B \text{ is infinite}$$
$$an\ even\ number\ of_M(A,B) \iff |A \cap B| \text{ is even}$$

Also,

(6) let **1** (**0**) be the trivially true (false) quantifier (*at least zero*, *fewer than zero*).

The following properties of quantifiers will be relevant:

(7) a. Q is *conservative* (Conserv) iff $Q_M(A,B) \Leftrightarrow Q_M(A, A \cap B)$

b. Q satisfies *extension* (Ext) iff for $A,B \subseteq M \subseteq M'$, $Q_M(A,B) \Leftrightarrow Q_{M'}(A,B)$

c. Q is *closed under isomorphism* (Isom) iff for $A,B \subseteq M$ and $A',B' \subseteq M'$, $|A-B| = |A'-B'|$, $|A \cap B| = |A' \cap B'|$, $|B-A| = |B'-A'|$, and $|M-(A \cup B)| = |M'-(A' \cup B')|$, entails $Q_M(A,B) \Leftrightarrow Q_{M'}(A',B')$

Call the first argument of a type $\langle 1,1 \rangle$ quantifier its *restriction* and the second its *scope*. Conserv and Ext together mean that quantification is in effect restricted to the restriction argument (Peters and Westerståhl, 2006, ch. 4.5). All quantifiers interpreting natural language determiners, in particular all those listed above, satisfy these two properties. Many satisfy Isom as well; in the list above, all do except those mentioning particular individuals, i.e. all except *John's* and *no _ except John*.

Ext entails that the universe is irrelevant, so we may drop the subscript $_M$ for such quantifiers, as we in effect did in Definition (4) of the various negations. One easily sees that the combination Conserv + Ext is *preserved* under inner and outer negation (and hence under duals),[5] and so is Isom. Also, note that Conserv + Ext entails that the definition of inner negation may be expressed as follows:

(8) $$Q\neg(A,B) \iff Q(A, A-B)$$

In what follows, we assume that these two properties hold of the quantifiers mentioned.

[5] But this fails for e.g. contrariety: it is easy find quantifiers Q and Q' which are contraries and such that Q satisfies Conserv + Ext but Q' doesn't.

2.2 Modern vs. classical squares

Every type $\langle 1,1\rangle$ quantifier *spans* a (modern) square. Define:

(9) $\quad square(Q) = \{Q, \neg Q, Q\neg, Q^{\mathrm{d}}\}$

The following facts are easily verified.

Fact 1

(a) $square(\mathbf{0}) = square(\mathbf{1}) = \{\mathbf{0}, \mathbf{1}\}$.

(b) *If Q is non-trivial, so are the other quantifiers in its square.*

(c) *If $Q' \in square(Q)$, then $square(Q) = square(Q')$.*

(d) *$square(Q)$ has either two or four members.*

By (c), any two squares are either identical or disjoint. As to (d), a square normally has four members, but it can happen that $Q = Q\neg$ (and thus $Q^{\mathrm{d}} = \neg Q$).

Example 2

This holds when Q expresses identical conditions on $A \cap B$ and $A - B$ (cf. (8) above), such as the following quantifier in the case when $k = n$,

$$Q_{(k,n)}(A,B) \iff |A-B| = n \;\&\; |A\cap B| = k$$

or the quantifier

$$\textit{exactly half}(A,B) \iff |A\cap B| = |A-B|$$

A more spectacular example is due to Ed Keenan, who noted that the sentences

(10) a. Between one-third and two-thirds of the students passed.
b. Between one-third and two-thirds of the students didn't pass.

are logically equivalent, i.e. that if Q = *between one-third and two-thirds of the*, then $Q = Q\neg$.[6]

Thus, applying these kinds of negation to type $\langle 1,1\rangle$ quantifier results in a rather robust 'square behavior'. And the main point here is that nothing similar holds for classical squares of opposition. To state this, we need to say what a classical square is. The following notion seems natural.

[6]See, for example, Keenan (2005). This observation is no mere curiosity, but a consequence of the following two general facts:

(i) a. (*at most p/q of the*)$\neg$ = *at least (q-p)/q of the* $(0 < p < q)$
b. $(Q_1 \wedge Q_2)\neg = Q_1\neg \wedge Q_2\neg$

Using this and the fact that $Q\neg\neg = Q$ one can verify that *between (q-p)/q and p/q of the* (for q-$p \leq p$) is identical to its inner negation. Note that the observation above about the quantifier *exactly half* is an instance of this.

(11) **Definition.** A *classical square* is an arrangement of four quantifiers as traditionally ordered and with the same logical relations – contradictories, contraries, subcontraries, and subalternates – between the respective positions.

Now each position determines the diagonally opposed quantifier, i.e. its outer negation, but *not* the quantifiers at the other two positions. For example, one may check the following:

Fact 3
The square

[**A**: *at least five*; **E**: *no*; **I**: *some*; **O**: *at most four*]

is classical. More generally, for $n \geq k$*,*

[**A**: *at least* n; **E**: *fewer than* k; **I**: *at least* k; **O**: *fewer than* n]

is classical.

The classical squares in Fact 3 look completely unnatural. There is no interesting sense, it seems, in which *no* is a negation of *at least five* or *at most four*. I conclude that the debate over issues pertaining to the Aristotelian square, such as those listed at the end of section 1.2, is to a large extent restricted to that particular configuration, and does not generalize to other quantifiers. In contrast, the modern square exhibits a completely general pattern of negation for type $\langle 1, 1 \rangle$ quantifiers, and thereby for the interpretation of simple and complex natural language determiners.[7]

3 Identifying the corners

In view of the preceding remarks, it looks like a mildly interesting task to investigate how various (modern) squares of opposition are 'manifested' in a language like English. In the next section I look at a few examples. But first we should deal with another matter.

While *square*(Q) uniquely specifies the *quantifiers* involved, it says nothing about how to distinguish the *corners*. Can we also find quantitative and qualitative aspects in the squares? Is it possible to identify **A**, **E**, **O**, and **I** corners in an arbitrary quantified square?[8]

[7]I am of course not claiming that these observations about negation and the square of opposition are new. The identification of the three forms of negation is made, for example, in Barwise and Cooper (1981) and Keenan and Stavi (1986) (as noted, Keenan calls inner negation *post-complement*), and it is fairly obvious that they generate squares of arbitrary quantifiers. Brown (1984) compares these to Aristotle's square, and Löbner (1990) also studies modern squares for other quantifiers (he calls them *duality squares*), although he disagrees with much of the generalized quantifier analysis of noun phrases. But it does seem to me that the behavior of modern squares of opposition in the context of natural language semantics has not been sufficiently explored in the literature. Exactly what I mean by this will be apparent in the following sections.

[8]Thanks to Larry Horn for directing my attention to this question.

In general, the answer is No. But in many cases we can obtain an identification, or at least a partial one, by suitably generalizing features of thoses corners in the Aristotelian square. The features I have in mind are purely semantic ones. Thus, I am not here attempting to say anything about which quantifiers in a square are more basic in terms of, say, conceptual or computational simplicity.

3.1 Monotonicity

A striking feature of the quantifiers in the Aristotelian square is their monotonicity behavior. Indeed, these quantifiers are *doubly monotone* (with a small *caveat* for the **A** and **O** corners).

(12)
a. Q is *right monotone increasing* (MON↑) iff
$Q(A,B)$ & $B \subseteq B' \Rightarrow Q(A,B')$

b. Q is *right monotone decreasing* (MON↓) iff
$Q(A,B)$ & $B' \subseteq B \Rightarrow Q(A,B')$

c. Q is *left monotone increasing* (↑MON) iff
$Q(A,B)$ & $A \subseteq A' \Rightarrow Q(A',B)$

d. Q is *left monotone decreasing* (↓MON) iff
$Q(A,B)$ & $A' \subseteq A \Rightarrow Q(A',B)$

Q is doubly monotone if it has both a left and a right monotonicity property. For example, *all* is ↓MON↑.[9]

Fact 4
(Peters and Westerståhl, 2006, ch. 5) *The monotonicity behavior of Q determines the monotonicity behavior of all elements of square(Q):*

1. *Q is* MON↑ *iff $Q\neg$ is* MON↓ *iff $\neg Q$ is* MON↓ *iff Q^d is* MON↑
2. *Q is* ↑MON *iff $Q\neg$ is* ↑MON *iff $\neg Q$ is* ↓MON *iff Q^d is* ↓MON
3. *So if Q is* doubly *monotone, all four combinations are exemplified in its square.*

This means that right monotonicity could be seen as *quality*, with MON↑ as *affirmative* and MON↓ as *negative*. Also (but less naturally), left monotonicity could be seen as *quantity*, with ↑MON as *particular* and ↓MON as *universal*. Thus, we can identify the exact position in the square of any doubly monotone quantifier.

However, many quantifiers are only *right monotone*, like the proportional quantifiers. So we can say, for example, that *at least two-thirds of the* is affirmative: it is either **A** or **I**, but we cannot say which. And this is not unreasonable: the dual of *at least two-thirds of the* is *more than one-third of the*, and it

[9] all_{ei} is MON↑ but only *weakly* ↓MON, in the sense that $Q(A,B)$ & $\emptyset \neq A' \subseteq A \Rightarrow Q(A',B)$. See section 4.4 for other weak monotonicity properties. As shown in (Peters and Westerståhl, 2006, ch. 5), the four properties listed above do not exhaust the monotonicity behavior of the quantifiers in Aristotle's square, but they are the only ones that will be used in this paper.

seems arbitrary which of these two should go into the **A** position. In any case, monotonicity cannot decide it (but see section 3.3 below).

We cannot *require* a certain monotonicity behavior of the quantifiers in *square*(Q), since many quantifiers are neither left nor right monotone, such as *an even number of* or *exactly ten* (though the latter is a conjunction of a ↑MON↑ and a ↓MON↓ quantifier). Rather, the rule should be that *when* Q exhibits monotonicity, the corners of *square*(Q) can be (partially) identified as described.

3.2 Symmetry

We can get some more help from a property that was identified and discussed already by Aristotle, who noted that the *order* between the two arguments of the quantifier is irrelevant at the **I** and **E** corners.

(13) a. Q is *symmetric* iff $Q(A,B) \Rightarrow Q(B,A)$.

b. Q is *co-symmetric* iff $Q\neg$ is symmetric.

Fact 5
(Peters and Westerståhl, 2006, ch. 6.1) *The symmetry behavior of* Q *determines the symmetry behavior of all elements of* square(Q)*:*

1. *Q is symmetric iff $Q\neg$ is co-symmetric iff $\neg Q$ symmetric iff Q^d is co-symmetric*
2. *Also, under* CONSERV *and* EXT*, symmetry is the same as* intersectivity*: if $A \cap B = A' \cap B'$ then $Q(A,B) \Leftrightarrow Q(A',B')$. So co-symmetry = co-intersectivity: if $A - B = A' - B'$ then $Q(A,B) \Leftrightarrow Q(A',B')$.*

Again, we cannot require that two quantifiers in *square*(Q) be (co-)symmetric, since e.g. squares of proportional quantifiers contain no symmetric quantifiers. But *when* there is symmetry behavior, Fact 5 says that we can use it to distinguish the **I** and **E** from the **A** and **O** corners.

Thus, if Q is *right monotone* and *either symmetric or co-symmetric*, then we can again pinpoint its exact position in the square, given that the **I** and **E** positions are symmetric, and the **A** and **O** positions co-symmetric. For example, *at most ten* is at the **E** position. But we already knew that, since *at most ten* is ↓MON↓. Indeed, if Q is right monotone and symmetric, it is clearly also left monotone. In fact, the cases where symmetry would give extra information are somewhat limited. This is also illustrated by the next result. Let FIN mean that attention is restricted to finite universes.[10]

Fact 6
(CONSERV, EXT, ISOM, FIN) *If Q is* MON↑*, and symmetric, then Q = at least n, for some $n \geq 0$.*

[10]Results like the next fact which rely on CONSERV, EXT, ISOM, and FIN are often easily proved with a 'number triangle argument'; see Peters and Westerståhl (2006) for explanations and several examples. I will not give these proofs here.

Thus, we only get extra information for cases like *an even number of*, which satisfies all assumptions of Fact 6 except monotonicity, and *no _ except John*, which is Conserv, Ext, and symmetric but not Isom or right monotone. This gives us two possible configurations of *square*(*an even number of*) and *square*(*no _ except John*).

We should also ask, however, if the two criteria for positioning quantifiers in squares, monotonicity and symmetry, can ever conflict with each other. After all, the intuitions behind them are rather different. A conflict would occur if we found a symmetric quantifier that was also either ↓Mon↑ or ↑Mon↓ (and correspondingly for co-symmetry). Fortunately, this cannot happen:

Fact 7
(Conserv, Ext) *If Q is symmetric and either* ↓Mon↑ *or* ↑Mon↓, *then Q is trivial* (*either* **0** *or* **1**).

Proof. Suppose Q is symmetric and ↓Mon↑. If $Q \neq \mathbf{0}$, there are A, B (by Ext, we don't have to worry about the universe) such that $Q(A,B)$ holds. We show:

(14) For all sets C, $Q(C,C)$.

This is enough, since then, for any sets D, E, we have $Q(D \cap E, D \cap E)$, hence $Q(D \cap E, D)$ by Conserv, so $Q(D, D \cap E)$ by symmetry, and so again by Conserv, $Q(D,E)$. This means that $Q = \mathbf{1}$.

To prove (14), take any set C. We have the following chain of implications: $Q(A,B) \Rightarrow Q(A \cap C, B)$ (by ↓Mon) $\Rightarrow Q(A \cap C, A \cap B \cap C)$ (by Conserv) $\Rightarrow Q(A \cap C, C)$ (by Mon↑) $\Rightarrow Q(C, A \cap C)$ (by symmetry) $\Rightarrow Q(C,C)$ (by Mon↑).

If Q is instead ↑Mon↓, then $\neg Q$ is ↓Mon↑ and symmetric (Facts 4 and 5), hence trivial by the above, and therefore so is Q. □

This is reassuring, and indicates that the intuitions behind identifying the corners by means of monotonicity and symmetry are quite robust.

Just as the selection of monotone and symmetric quantifiers is rather restricted (Fact 6), so is the choice of doubly monotone quantifiers, at least when Isom is presupposed. One can show the following (Westerståhl, 1989, sect. 4.3):

Theorem 8 (Conserv, Ext, Isom, Fin) ↑Mon↑ *quantifiers are finite disjunctions of quantifiers of the form* at least n of the k or more, *i.e.* $|A| \geq k$ & $|A \cap B| \geq n$ $(0 \leq n \leq k)$. *More generally,* ↑Mon *quantifiers are finite disjunctions of quantifiers of the form* $|A \cap B| \geq n$ & $|A - B| \geq k$.

However, there is an interesting class of non-Isom quantifiers with significant monotonicity properties: the possessives. For example, *at least five of John's* is ↑Mon↑, hence belongs to the **I** corner. And (*all of*) *John's* is Mon↑ and *weakly* ↓Mon (you can decrease A as long as something belonging to John remains; cf. note 9), so it goes in the **A** corner. On the other hand, *most of John's* is Mon↑ but not left monotone, so it is affirmative, but there seems to be no logical indication of whether it is **A** or **I**. We come back to this in section 4.4.

3.3 Summing up

In addition to monotonicity and symmetry, one can sometimes use a more pragmatic and loose criterion, at least as a rule of thumb for drawing squares: *square*(Q) should reduce to the (modern) Aristotelian square in *limiting cases*. What a limiting case is varies with the type of quantifier considered, but we will see several examples below where this notion is quite natural.

In sum, two semantic features of the Aristotelian square generalize to arbitrary quantifiers, and they can often be used to identify, at least partly, **A**, **E**, **O**, and **I** corners in quantified squares. Surprisingly many quantifiers in natural languages are monotone and/or symmetric.[11] Sometimes other criteria seem natural as well. On the other hand, the endeavor of finding unique properties of each corner in squares of opposition should not be taken too far, I think. There is no obvious semantic reason why each such corner should always be identifiable in this way. After all, *square*(*all*) is just one of infinitely many squares.

It is now high time to have a look at some examples of squares.

4 Examples

4.1 Numerical quantifiers

By a *numerical* quantifier I intend one of the form *at least n* ($n \geq 0$) or Boolean combinations (including inner negation) thereof, i.e. also, for example, *at most n*, *more than n*, *all but n*, *exactly n*, *between k and n*. Here is a typical square in this class, *square*(*at least six*) (Figure 5). In this diagram and the following ones, I use the convention that italics marks a quantifier interpreting a corresponding English determiner, so that one can see directly which corners of the square are 'realizable' as simple or complex determiners.

The quantifiers in Figure 5 are doubly monotone, so there is no question about the identification of the appropriate corners. In the limiting case, when *six* is replaced by *one* (or *five* by *zero*) we get *square*(*all*).

Before making some comments, let us look at another case (Figure 6): This time there is no monotonicity, but we have symmetry and co-symmetry, so *exactly five* should be either at the **E** or the **I** corner. The choice made in Figure 6 to place in at the **E** corner is dictated by the fact that with *five* replaced by *zero*, we again obtain *square*(*all*).

These squares are perhaps not very exciting, but there is nothing wrong with them. The truth conditions at each corner are clear, and one sees how English 'realizes' at least five of the eight corners by means of determiners. It is somewhat doubtful whether "all but at least six" is a well-formed English determiner; see Peters and Westerståhl (2006, p. 132) for some discussion. No determiners seem to correspond to the **I** and **O** corners of *square*(*exactly five*).

[11] See also ch. 5.5 of Peters and Westerståhl (2006), where it is shown that symmetry can in fact be seen as a monotonicity property too. So basically, it is in terms of monotonicity that I am trying to cash in the Aristotelian and medieval ideas (with the somewhat misleading labels) of quality and quantity.

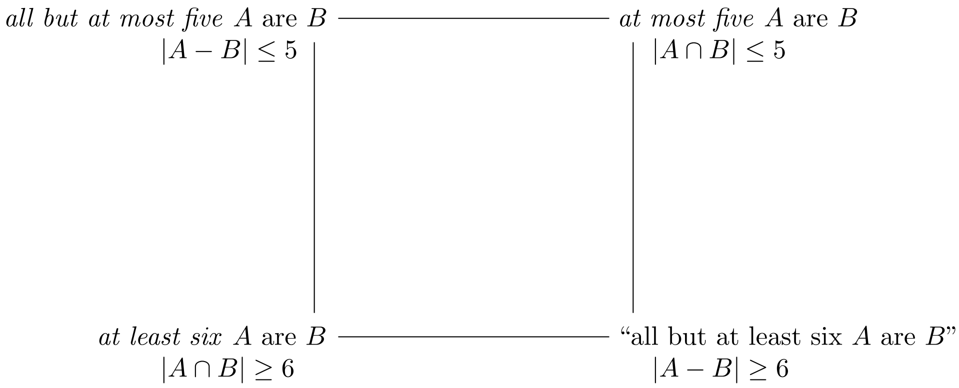


Figure 5: *square*(*at least six*)

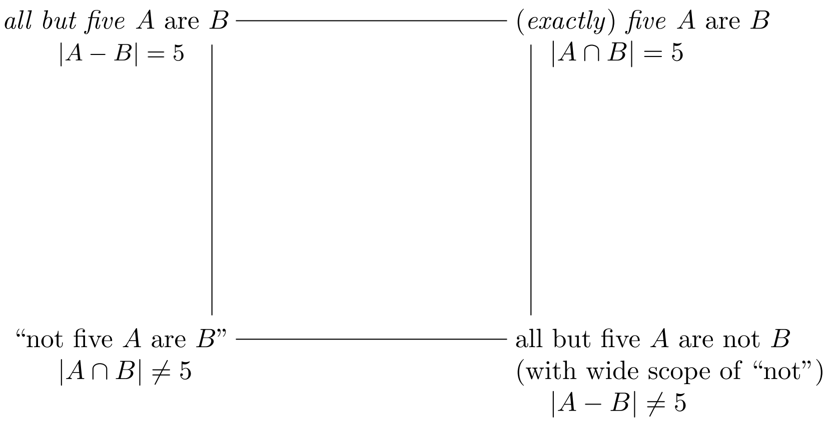


Figure 6: *square*(*exactly five*)

But the main point here is that these squares are far from classical. For example, $|A \cap B| \leq 5$ and $|A - B| \leq 5$ are compatible (provided $|A| \leq 10$), so they are not contraries. And $|A \cap B| = 5$ does not imply $|A - B| \neq 5$ (unless $|A| \neq 10$), etc. Will the squares become classical under suitable presuppositions, just as *square*(*all*) becomes classical if one presupposes that the restriction argument is non-empty? They will, but the presuppositions are not ones that anyone would deem reasonable in contexts where these expressions are used. This is seen from the next (easily verified) fact.

Fact 9
(a) *square*(at least $n+1$) *is classical iff* $|A| > 2n$ *is presupposed.*
(b) *square*(exactly n) *is classical iff* $|A| \neq 2n$ *is presupposed.*

Obviously, it makes no sense at all to have

(15) Five students passed the exam.

presuppose that the number of (salient) students was distinct from ten. *Exactly five* simply doesn't fit in a classical square of opposition.

4.2 Proportional quantifiers

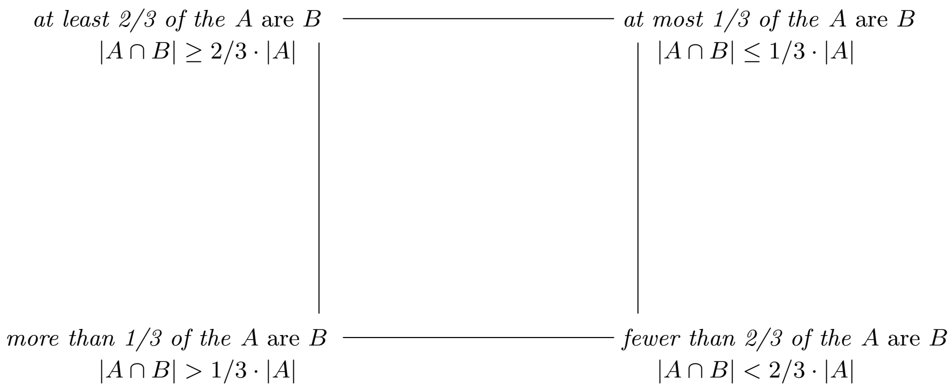


Figure 7: *square*(*at least 2/3 of the*)

In Figure 7, $Q =$ *at least 2/3 of the* is MON↑, but not left monotone or symmetric, so two configurations are possible: either Q or Q^{d} goes in the **A** corner. But it is natural to think of $p = q$ as a limiting case of *at least p/q of the*, and then $|A \cap B| \geq p/q \cdot |A|$ becomes $|A \cap B| \geq |A|$, i.e. $A \subseteq B$ (assuming FIN, which is reasonable for proportional quantifiers). So putting Q in the **A** corner, we again get *square*(*all*) in the limiting case.

We note that all four corners are 'realized' as English determiners, and that the square is not classical. However, in this case it is at least possible to argue that there is existential import at each corner, namely, if one gives a compositional analysis of determiners of the form [Det of the], and notes that *the* does have existential import.[12] But if $A \neq \emptyset$ is added to the truth conditions, we no longer have a (modern) square. (But we do get a classical one.) This could be taken to favor a presuppositional analysis of proportional quantifiers as regards existential import.

Alternatively, we can of course consider *square*(*at least 2/3 of the*$^+$), where *at least 2/3 of the*$^+(A, B) \Leftrightarrow |A \cap B| \geq 2/3 \cdot |A| > 0$. In this square (which incidentally is both classical and modern), we get the same problem at the **O** and **I** corners that the Aristotelian square had at the **O** corner, i.e. that the truth condition is a disjunction, one of whose disjuncts is $A = \emptyset$. The whole matter boils down to understanding exactly what happens when proportional determiners occur in negation contexts. This might be interesting to investigate, but I will not pursue it here.

4.3 Exceptive quantifiers

Exceptive quantifiers, i.e. quantifiers involved in the interpretation of exceptive noun phrases, like "every professor except Mary" or "No students except foreign exchange students", have been studied extensively in the literature, (see Peters and Westerståhl (2006, ch. 8) for an overview of the issues involved and a proposed general analysis). Their interaction with negation is not without interest. Here I just exemplify with the very simplest case (Figure 8).

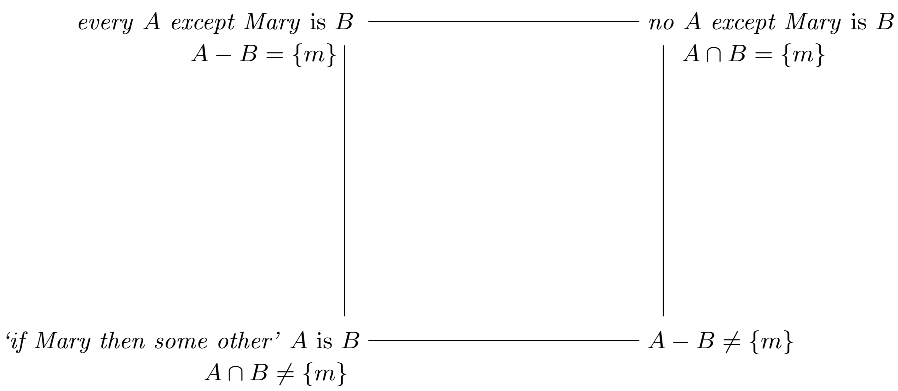


Figure 8: *square*(*every _ except Mary*)

In this square there is (co-)symmetry and no monotonicity. But in the limiting case when the exception set is *empty* we obtain *square*(*all*). *square*(*every _ except John*) is both modern and classical. The **O** corner appears to be unrealized. A suggestion (from Peters and Westerståhl (2006, ch. 4.3)) has been made

[12]Such such an analysis is given in Peters and Westerståhl (2006, ch. 7.11).

in Figure 8 for the **I** corner; it should be taken as *possible* English determiner with the desired interpretation.

4.4 Possessive quantifiers

The final example comes from possessive constructions. It relies on the account of possessives given in chapter 7 of Peters and Westerståhl (2006), and on further development of that work in Peters and Westerståhl (2010).[13] But my remarks here, which are meant only to illustrate how these quantifiers interact with negation, are mostly independent of the fine details of the semantics of possessives. I will just state the bare essentials needed to present the examples.

The main observation is that possessives, in prenominal as well as postnominal form, involve *two quantifiers*, one (Q_1) over the 'possessors' and the other (Q_2) over the 'possessions'. This holds for all possessives,[14] even though Q_2 is sometimes implicit. The following examples illustrate this.

(16) a. Mary's pupils are bright.
b. No car's tires were slashed.
c. Most of John's friends came to the party.
d. Some pupils of most teachers failed the exam.

In (16-d) both quantifiers are clearly visible: $Q_1 =$ *most* quantifies over teachers (the 'possessors') and $Q_2 =$ *some* quantifies over pupils (the 'possessions'). In (16-c), Q_2 (*most*) is again explicit; the 'possessor' is John, and we can think of *John* as a Montagovian style type $\langle 1 \rangle$ quantifier. But in (16-a) and (16-b), Q_2 is implicit. In (16-a) it is presumably *all*. But in (16-b) it is unlikely to be *all*; the sentence doesn't normally express that no cars had all of their tires slashed (which is consistent with lots of cars having a few tires slashed); rather, it says that no car had *any* tire slashed, i.e. $Q_2 =$ *some*.

This leads us to take the genitive suffix "'s" as well as the genitive preposition

[13] There is a vast literature on possessives, and it may seem strange that I have to refer to a relatively new account and not to some standard theory. But the fact is that this literature has been mostly concerned with the simplest cases of possessive constructions, and in particular not paid attention to their doubly quantificational nature, which is crucial here, but easily missed in the simplest cases. An exception, however, is Barker (1995), who notes that possessives involve quantification both over the possessors and the possessions. He attempts to handle both at the same time with a form of unselective binding à la Lewis (1975). Using two monadic quantifiers instead of one polyadic quantifier simplifies matters; the relation between Barker's account and ours is discussed in Peters and Westerståhl (2010).

[14] That is, if one disregards *modifying* or *descriptive* uses of the genitive. This use is most clear in compound noun-like constructions like "child's toys" or "men's room", but it can also occur with quantifiers, and there can be an ambiguity between the modifying use and what we call the *quantifying* use, as in

(i) Five students' tennis rackets were left in the locker room.

In the modifying use, *five* quantifies over tennis rackets: 'five tennis rackets of the kind used by students were left'. In the quantifying use, it quantifies over students: 'five students were such that their tennis rackets (which together may be more than five) were left'. I consider only quantifying uses here.

"of" to denote a higher-order operator *Poss*, which takes Q_1 and Q_2, as well as a set C and a binary relation R as arguments, and yields a type $\langle 1,1\rangle$ quantifier as output:

(17) Three athletes of each country paraded.
Q_2 A *Poss* Q_1 C B

Note that "Three of each country's athletes paraded" means exactly the same. The only part of the semantics which is not explicit in these sentences is the possessor relation R. That is precisely correct (we claim), since that relation is 'free' in the sense that even though a default relation exists in many cases, one can always think of contexts where a completely different relation would be intended. Even "John's brothers" can in a suitable context refer to some brothers that John is, say, responsible for, but who are not brothers of John. We thus leave R as a parameter in the semantics.

The definition of *Poss* is as follows:[15]

(18) $$Poss(Q_1,C,Q_2,R)(A,B) \iff Q_1(C \cap dom_A(R), \{a\colon Q_2(A \cap R_a, B)\})$$

I will not be concerned here with further motivation of this definition, just explain that R_a is the set of things 'possessed' by a, $\{b : R(a,b)\}$, and $dom_A(R)$ is the set of things 'possessing' something in A, $\{a : A \cap R_a \neq \emptyset\}$.

Now let us see how *Poss* behaves under negation. The following fact is easily verified.

Fact 10

(a) $\neg Poss(Q_1, C, Q_2, R) \;=\; Poss(\neg Q_1, C, Q_2, R)$

(b) $Poss(Q_1, C, Q_2, R)\neg \;=\; Poss(Q_1, C, Q_2\neg, R)$

(c) $Poss(Q_1, C, Q_2, R) \;=\; Poss(Q_1\neg, C, \neg Q_2, R)$

So we know what the inner and outer negation of $Poss(Q_1,C,Q_2,R)$ is, and thus its square of opposition. For example, for the universal reading of *Mary's*, i.e. the one with $Q_2 = all$, one verifies that according to (18),[16]

(19) $$Mary\text{'}s(A,B) \iff \emptyset \neq A \cap R_m \subseteq B$$

where m = Mary. In other words, "Mary's As are B" says that Mary 'possesses' at least one A (a kind of possessive existential import), and all of the As she 'possesses' are B. We get the square of Figure 9.

We see from (19) that *Mary's* is Mon↑ and *weakly* ↓Mon (you can decrease A as long as $A \cap R_m \neq \emptyset$), so it should be in the **A** corner. Likewise, at the

[15] It can be shown that when Q_1 and Q_2 are Conserv and Ext, so is $Poss(Q_1,C,Q_2,R)$, which is why the universe M has been left out in (18).

[16] Actually, (18) is formulated for a type $\langle 1,1\rangle$ Q_1 with a 'frozen' noun argument C, as in *each country*, but *Mary* is a type $\langle 1\rangle$ quantifier I_m (defined by $I_m(B) \Leftrightarrow m \in B$). So to apply (18) we need to rewrite I_m as a 'frozen' type $\langle 1,1\rangle$ quantifier; this can be done, for example, with $Q_1 = all_{\mathrm{ei}}$ and $C = \{m\}$. In Peters and Westerståhl (2006, ch. 7) we argue that this apparently *ad hoc* move (a) gives correct truth conditions in all cases where it is needed, and (b) is necessary if one wants a unified account of possessives.

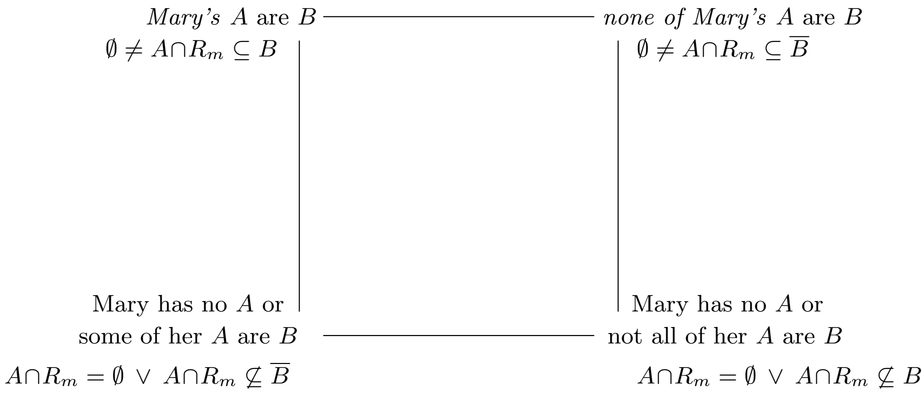


Figure 9: *square*(*Mary's*)

E corner we have MON↓ and weak ↓MON. But because these left downward monotonicities are weak, we do not have double monotonicity at the remaining two corners. More exactly, the **I** corner is MON↑ but not ↑MON (since we can have $A \cap R_m = \emptyset$, $A \subseteq A'$, and $\emptyset \neq A' \cap R_m \subseteq \overline{B}$), and the **O** corner is MON↓ but not ↑MON. But the existing monotonicities suffice to uniquely name the corners of *square*(*Mary's*).

Now, while in this case the **E** corner provides a natural kind of negation of "Mary's As are B", the **O** corner does not, and the **I** corner too seems odd.[17] That is, it is unlikely that

(20) Mary's pupils aren't bright.

could mean that *either* Mary has no pupils *or* some of her pupils are not bright. Even for the logician's

(21) It is not the case that Mary's pupils are bright.

one would be hard pressed to get that reading. Still, (20) is ambiguous. It can mean what the inner negation yields, that she has pupils but none of them are bright. But it also seems possible to use (20) to express that she has pupils and some of them aren't bright. This becomes clearer if we start instead from a version of the positive statement, i.e. the universal reading of (16-a), where Q_2 is explicit:

(22) All of Mary's pupils are bright.

Now it is rather clear that

[17]Note that the **I** corner of a square, i.e. the dual of the **A** corner, should not be seen as a form of negation on a par with the **E** and **O** corners. It is rather a kind of *double* negation. Still, it is often 'realized' in English, but the truth condition at that corner in Figure 9 seems distinctly odd.

(23) All of Mary's pupils aren't bright.

has both readings (though this time with a preference for the second one), whereas

(24) Not all of Mary's pupils are bright.

seems to have only the second reading. But none of these negative statements expresses the truth conditions of the **O** corner.

It might seem natural to deal with this situation similarly to a strategy hinted at for proportional quantifiers earlier, i.e. to always presuppose 'possessive' existential import, so that at all corners in Figure 9, $A \cap R_m \neq \emptyset$ is assumed. But this overlooks two important facts. First, although contradictory negation doesn't seem to occur for (16-a), it occurs for other possessives. For example, consider

(25) Not everyone's needs can be satisfied with standard products.

This seems simply to deny that everyone is such that (all) his/her needs can be satisfied with standard products, i.e. it says that someone has at least one need that cannot be so satisfied. Note that when *everyone's* is analyzed with *Poss* according to (18), there is no 'possessive' existential import, because Q_1 = *every* (on our account) doesn't have existential import. And indeed it seems that (25) can be truly uttered in the case when no one in fact has any needs (even if it would be odd to for someone to utter it knowing that).

The second, perhaps even more compelling, fact is that our semantics using *Poss* provides a straightforward way of expressing the desired truth conditions. It is just a new form of negation, applicable in cases when a quantifier is composed out of two other quantifiers as in *Poss*. Call this *middle negation*:

(26) $\neg^m Poss(Q_1, C, Q_2, R) \;=_{\mathrm{def}}\; Poss(Q_1, C, \neg Q_2, R)$

(By Fact 10 (c) and obvious laws of negation we also have $\neg^m Poss(Q_1, C, Q_2, R) = Poss(Q_1\neg, C, Q_2, R)$.) There is a corresponding 'middle dual':

(27) $Poss(Q_1, C, Q_2, R)^{\mathrm{d}^m} \;=_{\mathrm{def}}\; Poss(Q_1, C, (Q_2)^{\mathrm{d}}, R)$

We now have a 'square of middle opposition':

(28) $square^m(Q) \;=_{\mathrm{def}}\; \{Q, Q\neg, \neg^m Q, Q^{\mathrm{d}^m}\}$

This square is easily seen to have the same crucial square property as the standard one:

Fact 11
If Q' belongs to $square^m(Q)$, then $square^m(Q') = square^m(Q)$.

For *Mary's*, we can now check that the relations in the diagram of Figure 10 obtain. Here all four corners are realized as possessive determiners, and we have in one diagram the two natural ways to negate (16-a). Further, the

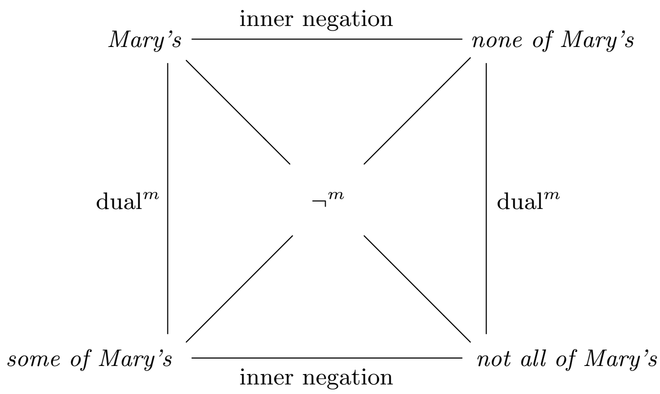


Figure 10: $square^m(Mary's)$

semantics itself gives all corners 'possessive' existential import, and we have double monotonicity of the expected kind at each corner.

But we shouldn't stop there. Ordinary contradictory negation and ordinary dual also exist for possessive quantifiers. Moreover, $square(Q)$ and $square^m(Q)$ are closely related. In fact, they represent all the possible ways to negate quantifiers of this form. In principle, considering the four standard possibilites for Q_1, i.e. the four members of $square(Q_1)$, and similarly for Q_2, there would be 16 corresponding cases for $Poss(Q_1, C, Q_2, R)$. But, using Fact 10 (c) and standard laws of negation, one sees that

Fact 12
Of the 16 cases for putting, or not, inner and outer negations and duals in $Poss(Q_1, C, Q_2, R)$*, only 8 yield distinct quantifiers.*

Furthermore, these 8 quantifiers are naturally represented in a *cube of opposition*, that we can define by means of the diagram in Figure 11, for $Q = Poss(Q_1, C, Q_2, R)$. Here, the front side of the cube is $square(Q)$, and the top side is $square^m(Q)$. Also, the back side is $square(\neg^m Q)$, and the bottom side is $square^m(\neg Q)$. This means that the four horizontal edges of the cube (oriented as in the diagram) stand for inner negation, the four vertical edges stand for dual, and the four remaining, slanted, edges for middle dual. Likewise, the diagonals of the front and back faces represent outer negation, and those on the top and bottom faces middle negation. The remaining diagonals, those on the remaining sides and the four diagonals inside the cube, also stand for easily identifiable operations on quantifiers of the form $Poss(Q_1, C, Q_2, R)$.

The robustness of this representation is indicated by the following fact.

Fact 13
(a) *If* $Q' \in cube(Q)$*, then* $cube(Q') = cube(Q)$*.*

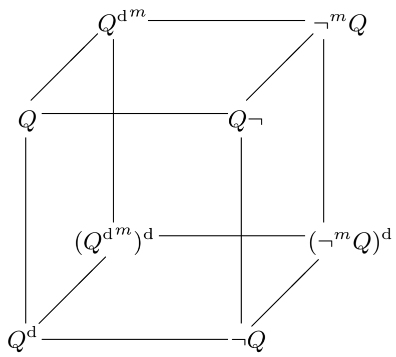


Figure 11: *cube*(Q)

(b) *The same 'cubic behavior' is exhibited by any binary operation F from quantifiers to quantifiers satisfying the conditions in Fact 10:*

- $\neg F(Q_1, Q_2) \;=\; F(\neg Q_1, Q_2)$
- $F(Q_1, Q_2)\neg \;=\; F(Q_1, Q_2\neg)$
- $F(Q_1, Q_2) \;=\; F(Q_1\neg, \neg Q_2)$[18]

Can we distinguish corners also in the cube? Continuing to use monotonicity as a guide, and following the example *Mary's*, we calculate:

Fact 14
The monotonicity behavior of cube(Mary's)*, oriented as in Figure 11 with $Q =$* Mary's*, is as depicted in Figure 12.*

Interestingly, the monotonicity facts for the two ordinary squares, i.e. the front and back sides, of this cube are similar but not quite the same. At the front (*square*(*Mary's*)), we only have weak left monotonicity at the **A** and **E** corners, and consequently only right monotonicity at the **O** and **I** corners. At the back (*square*($\neg^m$*Mary's*)), on the other hand, we have full double monotonicity at each corner. Still, this is enough to uniquely name each corner as we have

[18] Another example of an operation of the kind mentioned in (b), very different from *Poss*, is *iteration*, i.e. the natural way to combine the two quantifiers occurring in simple sentences with a quantified subject and object (see, for example, Peters and Westerståhl (2006, ch. 10.1)). The operation *It* takes two type $\langle 1\rangle$ quantifiers and yields a type $\langle 2\rangle$ quantifier (or two type $\langle 1,1\rangle$ quantifiers yielding a type $\langle 1,1,2\rangle$ quantifier), so that, for example, the interpretation of

(i) At least two critics reviewed four films.

is obtained by iterating the type $\langle 1\rangle$ quantifiers $Q_1 =$ *at least two critics* and $Q_2 =$ *four films*, and applying $It(Q_1, Q_2)$ to the relation *reviewed*. I am not saying that the corresponding cube of opposition has the same interest as in the possessive case, but the formal similarity between the ways these operators interact with negation is at least striking.

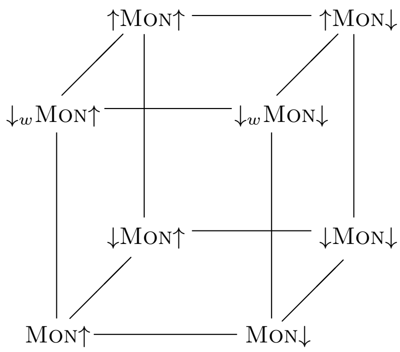


Figure 12: Monotonicity in *cube*(*Mary's*) ($_w$ means 'weak')

done for ordinary squares. In the cube, then, we will have *two* **A** corners, *two* **E** corners, etc. Specifically, at the back side we have, going clockwise and starting from $\neg^m$*Mary's*: **O**, **E**, **A**, **I**. Thus, the square in the back is 'upside down' compared to the one at the front; see Figure 13. Note that this is also

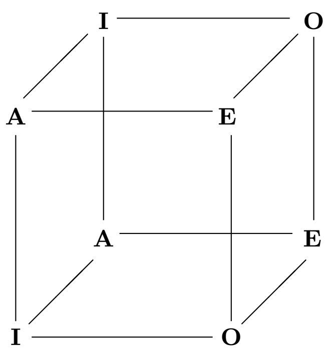


Figure 13: Corners in *cube*(*Mary's*)

consistent with the naming suggested earlier for the corners of *square*^{m}(*Mary's*) (and *square*^{m}($\neg$*Mary's*)).

Another thing emerging from the examples discussed so far is that the left monotonicity behavior of a possessive quantifier Q does *not* determine that of the other quantifiers in *square*^{m}(Q), in contrast with the situation for ordinary squares. More precisely, we have the following:

Fact 15
If Q is a possessive quantifier, then Q is Mon↑ *iff $Q\neg$ is* Mon↓ *iff $\neg^m Q$ is* Mon↓ *iff Q^{d^m} is* Mon↑. *But one cannot, without extra information, draw any conclusions from the left monotonicity of Q, except for $Q\neg$ as described in Fact 4.*

Proof. The positive claims are easily verified directly from definition (18). The negative claims are illustrated by the top and bottom sides in Figure 12. □

In conclusion, let us look at one more possessive example. We already noted that in sentence (16-b), *no car's* normally has an existential reading ($Q_2 =$ *some*). Other sentences with *no C's* can be ambiguous:

(29) No students' library books were returned in time.

This can mean that no student returned *any* of his/her books in time, for example, as uttered by someone who then concludes that the school needs to find stronger inducements for students to return at least some library books on time. But it could also have the universal reading that no student returned *all* of his/her books in time, e.g. as said by a librarian noting that every student who checked out library books had to pay at least one late fee.[19] Not surprisingly, both these readings occur in the same cube. Instead of drawing the whole cube with all the information, I content myself with the ordinary and the middle squares of the universal reading of *each car's* (equivalently, the existential reading of *no car's*): Figures 14 and 15. The truth conditions are all calculated from the definition of *Poss* in (18). This time, there is no 'possessive' existential

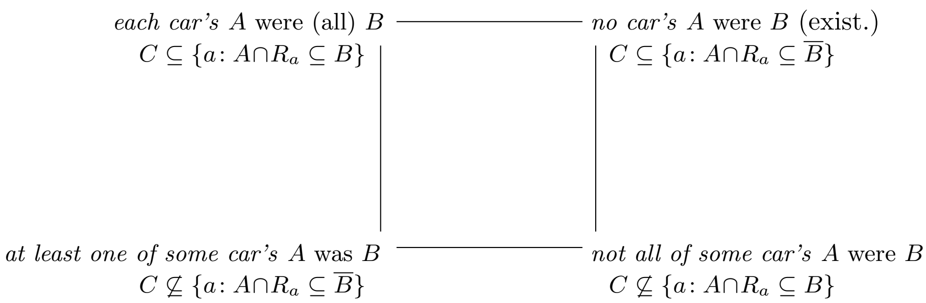


Figure 14: *square*(*each car's*) (universal reading)

import. One sees that *each C's* is ↓Mon↑, and so we have double monotonicity at each corner (Fact 4) in Figure 14. But, as indicated in Fact 15, this does not hold in Figure 15. The **I** and **O** corners can still be uniquely identified in this diagram, since we have Mon↑ and Mon↓ there, respectively, *and* since the **A**

[19]The example is from Barbara Partee (p.c.).

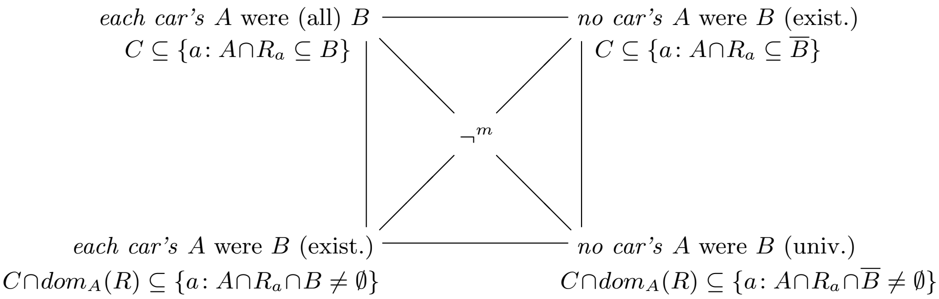


Figure 15: $square^m$(*each car's*) (universal reading)

and **E** corners were already identified, but one can verify that left monotonicity fails at these corners. The universal reading of *no C's*, as in one interpretation of (29), is found at the **O** corner of $square^m$(*each car's*).

Thus, we have identified all eight corners of *cube*(*each car's*) (universal reading). To find the truth conditions at the two corners not present in Figure 14 or 15, recall that the back side of this cube is an ordinary square with outer negation along its diagonals. Using this, Figure 15, and Fact 10 (c), one sees that the remaining **A** corner expresses the universal reading of *some car's* A were B, and the remaining **E** corner expresses that *none of some car's* A were B. So we also see that, due to the flexibility of the formation rules for English possessive determiners, each corner of the cube is in this case 'realized' by such a determiner.

However, the monotonicity behavior of possessive quantifiers is somewhat complex (Peters and Westerståhl, 2006, ch. 7.13), and we cannot in general expect all corners of cubes from possessive quantifiers to be uniquely identifiable in terms of monotonicity, even when both component quantifiers Q_1 and Q_2 are monotone.

5 Concluding remarks

The square of opposition is a useful conceptual tool for understanding how negation interacts with quantification (and thereby several other operators). It doesn't represent a particular quadruple of quantifiers, such as the four Aristotelian ones, but rather a pattern that recurs for *all* quantifiers. This pattern, I have argued, uses the modern square, with the two basic forms of negation, and their combination with each other, the dual. Characteristically, further combination of these three operators yields nothing new; it stays within the square.

I have not discussed which of the three operators is most basic. But such a discussion, if one wants to embark on it, should start from the general square pattern, not from any particular square of opposition. We defined dual in terms of inner and outer negation, but one can take any two of the three operators and define the third in terms of them: $Q\neg = \neg(Q^{\mathrm{d}})$ and $\neg Q = (Q^{\mathrm{d}})\neg$. One cannot define any of the operators, seen as operators on quantifiers, in terms of just one of the others (they are all idempotent). In this sense, they are all on a par. But other notions of definability can perhaps be invoked; I leave to others to ponder whether this is a worthwhile idea.[20]

In this paper I have, aside from a few general observations, used examples to illustrate the variety of negation-quantification interaction in natural language. In particular, complex determiners built from simpler ones, such as the possessive determiners, constitute a rich source of examples, a source it may be worth exploring further.

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[20] If pressed, I would argue that *outer* negation is most basic, along the following (loose) lines: Outer negation is sentence negation. Inner negation is also sentence negation, though applied to an 'interior' sentence. Dual combines the two.

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