Definability of Polyadic Lifts of Generalized Quantifiers

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Abstract

We study generalized quantifiers on finite structures. With every function $f : \omega \to \omega$ we associate a quantifier Q_f by letting $Q_f x\varphi$ say "there are at least $f(n)$ elements x satisfying φ, where n is the size of the universe". This is the general form of what is known as a monotone quantifier of type $\langle 1 \rangle$. We study so called *polyadic lifts* of such quantifiers. The particular lifts we consider are Ramseyfication, branching and resumption. In each case we get exact criteria for definability of the lift in terms of simpler quantifiers.

1 Introduction and preliminaries

Monadic generalized quantifiers express properties of isomorphism types of monadic structures, or equivalently, relations between cardinal numbers. Ar-

*Partially supported by a grant from the University of Helsinki. This research was initiated while the first author was a Junior Researcher at the Academy of Finland.

†Partially supported by grant 1011040 from the Academy of Finland and a grant from the University of Helsinki.

‡Supported by the Swedish Council for Research in the Humanities and Social Sciences, and by the Academy of Finland (during a visit to the University of Helsinki).

bitrary (polyadic) generalized quantifiers, however, express properties of isomorphism types of arbitrary relational structures. Apart from the issue of the adequacy of the term 'quantifier' here — only monadic generalized quantifiers deal with *quantities* — the step from monadic to polyadic quantifiers entails a marked increase in difficulty as to issues of expressive power.

Generalized quantifiers are added to first-order logic — or to some other logic — as a means of increasing its expressive power in certain respects (you might want to be able to talk about infinity, or well-order, or compare cardinals, in your logic). But precisely how expressive is the new logic? This leads to basic definability questions like: When is quantifier Q definable in the logic obtained by adding quantifiers $Q_1, \ldots, Q_n$ to a given logic? Such questions can be quite hard. On finite models, answering some of these questions would mean solving notoriously difficult open problems about computational complexity.

The results on (un)definability in the literature usually concern particular quantifiers; few general definability results for interesting classes of quantifiers are known. For monadic quantifiers, however, the definability question above has recently been given a complete answer in algebraic terms (cf. [19]), an answer which also permits complete classifications with respect to expressive power of various classes of monadic quantifiers.

Nothing similar is in sight for polyadic quantifiers, not even for the simplest kind whose signature consists of one binary relation. A rough way of measuring complexity of a quantifier is by its *arity*, i.e., the largest arity of the relations occurring in the corresponding signature. So monadic quantifiers are unary, and an early result by Lindström [14] is that the binary well-ordering quantifier is not definable by means of any monadic quantifiers added to first-order logic. This leads to the question of the existence of *arity hierarchies* of quantifiers: sequences $\langle Q_n \rangle_{n \in \omega}$ where Q_n is n-ary and not definable in terms of Q_k for $k < n$, or even in terms of any quantifiers of lower arity. Several such hierarchies are known; a systematic treatment is given in [6].

These hierarchy results make essential use of infinite models. Recently, some hierarchy results have also been obtained for finite models. One result by Luosto [15] is that the relativization of the binary quantifier "R contains at least half of the ordered pairs of elements of the universe" is not definable from any monadic quantifiers over finite models. The proof uses advanced finite combinatorics. Another recent result, the Hierarchy Theorem of [8], states that, with respect to a more fine-grained complexity ordering than the arity ordering, there exists at every step quantifiers not definable in terms of quantifiers of lower complexity, over finite models. However, the existence of these quantifiers is proved by probabilistic methods, not by explicit construction.

In this paper we shall give definability characterizations over finite models for certain classes of polyadic quantifiers, more precisely for polyadic quantifiers that are obtained from monadic ones by means of certain operations. We call these operations *polyadic lifts*. Three such lifts are studied: *branching, Ramseyfication* and *resumption*.

Let Q be a *simple unary* quantifier (one whose signature contains just one unary predicate symbol), which is also monotone. Then there is a function f from natural numbers to natural numbers such that on a universe with n elements Q means "at least $f(n)$". The *branching* of Q (with itself) says of a binary relation R that there are subsets X, Y of the universe with at least $f(n)$ elements such that the 'rectangle' $X \times Y$ is contained in R. Similarly we can branch two distinct monotone simple unary quantifiers (requiring then that $|X| \geq f(n)$ and $|Y| \geq g(n)$), or k such quantifiers. The k-ary *Ramseyfication* of Q says that there is a set X of cardinality at least $f(n)$ which is homogeneous for R, i.e., any k distinct elements of X stand in the relation R to each other. A *resumption* of a monadic quantifier, finally, says the same thing about tuples of individuals that the monadic quantifier says about individuals. So the k-ary resumption of Q says, on a universe with n elements, of a k-place relation that it contains at least $f(n^k)$ k-tuples.

These polyadic lifts turn up in various contexts. The Ramseyfication operation is familiar from model theory. The ability to quantify over k-tuples is sometimes assumed in finite model theory in order to be able to describe certain behavior of Turing machines. And all three lifts have been proposed in natural language semantics. Actually, this is the context where the idea of polyadic lifts and their definability arose, even though the definability issues themselves are standard mathematical-logical questions.

The most immediate question to ask about a polyadic lift is whether it really gives anything new, i.e.: Is *Lift*$(Q_1, \ldots, Q_n)$ definable in terms of $Q_1, \ldots, Q_n$? This is the question we deal with in this paper. We prove that for the three lifts under consideration, the lifted quantifiers are usually not definable in first-order logic augmented with the argument quantifiers. More precisely, we show that a certain condition of unboundedness of the functions associated with monotone simple unary quantifiers is sufficient for undefinability of branching and Ramseyfication. Actually, our results are stronger. We show that, if Q is unbounded, the $(k+1)$-ary branching of Q is not definable from *any* monadic quantifiers added to first-order logic ($k \geq 1$). And for Ramseyfication we obtain undefinability even if any quantifiers of arity at most k are allowed, and even if first-order logic is replaced by the stronger logic $\mathcal{L}^{\omega}_{\infty\omega}$ (a logic which extends fixed point logic). Moreover, we also show that unboundedness is a necessary condition for undefinability. That is, if Q is bounded, branching and Ramseyfication of Q *is* definable in terms of Q. We have a similar characterization for the definability of the branching of $k+1$ quantifiers.

For resumption, our results are similar but a little weaker. We show that a slight strengthening of unboundedness is necessary and sufficient for the $(k+1)$-ary resumption of Q to be undefinable from any monotone simple unary quantifiers added to first-order logic. We also identify another necessary condition for the $(k+1)$-ary resumption of Q to be undefinable from Q, roughly that the behavior of Q on universes of size n does not determine its behavior on universes of size n^{k+1}.

In natural language semantics, the basic quantifiers are not the simple unary ones but instead those of type $\langle 1, 1\rangle$, whose signature has two unary predicate symbols. English determiners like *all, some, no, at least five, most, all but three, more than two thirds of*, etc., denote such quantifiers. The lift operations arise from the way noun phrases containing determiners are combined with each other in sentences. The standard lifts here give polyadic quantifiers which are definable from the determiner denotations, but some sentences appear to use the three lifts studied in this paper. We show that our results about branching and Ramseyfication generalize to type $\langle 1, 1\rangle$ quantifiers, provided these satisfy two conditions. These conditions are believed to hold (almost) universally for determiner denotations.

We restrict attention to finite models, a constraint which is common in natural language semantics and model theory for computer science. The *un*definability results of course hold for arbitrary models — it's just that the counter-examples we give to definability are finite. But our results that certain quantifiers are definable in certain logics require the assumption of finiteness.

In the remainder of this section we first recall the relevant notions pertaining to logics with generalized quantifiers, and then introduce the three lift operations in the context of natural language semantics. The brief linguistic excursion there is included for background and motivation, but not used in the rest of the paper. In Section 2 we state some facts about monotone quantifiers and establish that boundedness implies definability (of branching and Ramseyfication). Sections 3,4, and 5 deal with branching, Ramseyfication and resumption, respectively. In each case, undefinability is established by means of a suitable Ehrenfeucht-Fraïssé game, although the three proofs are quite different. Section 6 concludes the paper with some open questions.

1.1 Generalized quantifiers and definability

We begin by recalling the concept of a generalized (Lindström) quantifier, and of logics with such quantifiers (cf. [13]).

A (generalized) *quantifier* is a class Q of structures of a finite relational signature which is closed under isomorphism. The *type* of Q can be identified with a finite sequence $\langle k_1, \ldots, k_n \rangle$ of positive natural numbers ($n > 0$). Equivalently, Q can be defined as a functional assigning to each non-empty set A a *quantifier* Q_A *of type* $\langle k_1, \ldots, k_n \rangle$ *on* A, i.e., a subset of $\mathcal{P}(A^{k_1}) \times \ldots \times \mathcal{P}(A^{k_n})$ (a second-order relation over A). Instead of $(A, R_1, \ldots, R_n) \in Q$ we then write $Q_A(R_1, \ldots, R_n)$ or simply $Q_A R_1 \ldots R_n$. The *arity* of Q is $\max\{k_1, \ldots, k_n\}$. Q is *monadic* if it is unary ($k_1 = \ldots = k_n = 1$); otherwise *polyadic*. We let $\mathbf{Q}_k$ be the class of all quantifiers of arity $\leq k$.

A quantifier Q of type $\langle k \rangle$ is called *simple*. It is *monotone* if for all A and all $R, S \subseteq A^k$, $Q_A R$ and $R \subseteq S$ implies $Q_A S$. The notion of a type $\langle k_1, \ldots, k_n \rangle$ quantifier being *monotone in the i:th argument* is defined similarly.

With many familiar logics $\mathcal{L}$ one can uniquely associate a logic $\mathcal{L}(Q)$ obtained by adding the quantifier Q: a new formation rule yields formulas

$$Q\mathbf{x}_1, \ldots, \mathbf{x}_n(\varphi_1, \ldots, \varphi_n)$$

where $\varphi_1, \ldots, \varphi_n$ are formulas, $\mathbf{x}_i$ is a string of k_i distinct variables, and the semantics is given by the clause

$$\mathbf{A} \models Q\mathbf{x}_1, \ldots, \mathbf{x}_n(\varphi_1(\mathbf{a}, \mathbf{x}_1), \ldots, \varphi_n(\mathbf{a}, \mathbf{x}_n)) \iff (A, \varphi_1^{\mathbf{A}}(\mathbf{a}, \cdot), \ldots, \varphi_n^{\mathbf{A}}(\mathbf{a}, \cdot)) \in Q.$$

Here A is the universe of $\mathbf{A}$, $\mathbf{a}$ is a finite sequence of elements in A, and $\varphi_i^{\mathbf{A}}(\mathbf{a}, \cdot) = \{\mathbf{b} \in A^{k_i} \mid \mathbf{A} \models \varphi_i(\mathbf{a}, \mathbf{b})\}$. Similarly one defines the logic $\mathcal{L}(\mathbf{Q})$ when $\mathbf{Q}$ is a class of quantifiers.

In this paper $\mathcal{L}$ will be either first-order logic $\mathcal{L}_{\omega\omega}$, or $\mathcal{L}^k_{\infty\omega}$ (which is like $\mathcal{L}_{\infty\omega}$ except that it only has k variables), or $\mathcal{L}^\omega_{\infty\omega} = \bigcup_{k<\omega} \mathcal{L}^k_{\infty\omega}$. As usual,

$$\mathbf{A} \equiv_{\mathcal{L}(\mathbf{Q})} \mathbf{B}$$

means that the same $\mathcal{L}(\mathbf{Q})$-sentences are true in $\mathbf{A}$ and in $\mathbf{B}$. When φ is an $\mathcal{L}_{\omega\omega}(\mathbf{Q})$-formula, its *quantifier rank* $qr(\varphi)$ is the maximal number of nestings of quantifier symbols (including $\forall$ and $\exists$) in φ, and

$$\mathbf{A} \equiv^q_{\mathcal{L}_{\omega\omega}(\mathbf{Q})} \mathbf{B}$$

means that the same $\mathcal{L}_{\omega\omega}(\mathbf{Q})$-sentences of quantifier rank at most q are true in $\mathbf{A}$ and in $\mathbf{B}$.

The *relativization* of a type $\langle k_1, \ldots, k_n \rangle$ quantifier Q is the type $\langle 1, k_1, \ldots, k_n \rangle$ quantifier Q^{rel} defined by

$$Q^{\mathrm{rel}}_A(X, R_1, \ldots, R_n) \iff Q_X(R_1 \cap X^{k_1}, \ldots, R_n \cap X^{k_n}).$$

This can be extended to formulas of $\mathcal{L}(Q)$: if P is a (new) unary predicate symbol and φ such a formula one defines inductively a formula $\varphi^{(P)}$ of $\mathcal{L}(Q^{\mathrm{rel}})$ which says the same thing about $(\mathbf{A}, X)$ as φ says about the substructure of $\mathbf{A}$ generated by X.

The quantifier Q is *definable* in $\mathcal{L}(\mathbf{Q})$ if the sentence

$$Q\mathbf{x}_1, \ldots, \mathbf{x}_n(P_1(\mathbf{x}_1), \ldots, P_n(\mathbf{x}_n))$$

is logically equivalent to some $\mathcal{L}(\mathbf{Q})$-sentence in those predicate symbols. A logic *extends* another logic if every sentence in the latter is equivalent to some sentence in the former. In particular, $\mathcal{L}(\mathbf{Q}')$ extends $\mathcal{L}(\mathbf{Q})$ iff each $Q \in \mathbf{Q}$ is definable in $\mathcal{L}(\mathbf{Q}')$.

1.2 Polyadic lifts and natural language quantification

Before defining the polyadic lifts we briefly explain how they turn up in natural language semantics. More details can be found in [10].

Monadic generalized quantifiers provide the most straightforward way to give the semantics of *noun phrases* (NPs) in natural languages. Consider a sentence like

(1) No students smoke

It consists of a noun phrase *no students*, in turn constituted by the determiner *no* and the noun *students*, and a verb phrase *smoke*, in this case a simple intransitive verb. In a model $\mathbf{A}$ the noun and the verb phrase denote subsets of A. Thus, the determiner is naturally taken to denote a binary relation between subsets of A, i.e., with varying A, a type $\langle 1,1\rangle$ generalized quantifier.

In (1) the quantifier used is $no_A XY \Leftrightarrow X \cap Y = \emptyset$ (using the same symbol for the determiner as for its denotation). Changing the determiner in (1) we obtain other familiar type $\langle 1,1\rangle$ quantifiers such as *all, some, at least five, most, all but three, more than two thirds of,* etc., where, for example, $most_A XY \Leftrightarrow |X \cap Y| > |X - Y|$, and $all\ but\ three_A XY \Leftrightarrow |X - Y| = 3$.

The first argument in these quantifiers is called the *noun argument*, and the second the *verb argument*. Some determiners take more than one noun argument, as in

(2) More students than teachers smoke

where *more... than* denotes the type $\langle 1,1,1\rangle$ quantifier $more\ldots than_A XYZ \Leftrightarrow |X \cap Z| > |Y \cap Z|$. The role of type $\langle 1\rangle$ quantifiers in this context is as NP denotations. For example, the denotation of *most students* is obtained by fixing the noun argument of *most* to the denotation of *students*. Other NPs do not involve determiners, like *something* and *everything* which denote $\exists$ ($\exists_A X \Leftrightarrow X \neq \emptyset$) and $\forall$ ($\forall_A X \Leftrightarrow X = A$), respectively, and phrases like *John or Mary*, which denotes the set of $X \subseteq A$ such that $j \in X \vee m \in X$. But note that whereas the determiner denotations above are all closed under isomorphism, most NP denotations are not, and thus do not qualify as quantifiers in the present sense.

Determiners usually denote type $\langle 1,1\rangle$ quantifiers, but not all such quantifiers are determiner denotations. Indeed, the following two constraints have been found to hold (almost) universally:

CONS	$Q_A XY$	$\Longleftrightarrow$	$Q_A X\, X{\cap}Y$	
EXT	$Q_A XY$	$\Longleftrightarrow$	$Q_{A'} XY$	for $X, Y \subseteq A, A'$

CONS predicts, for example, that the *Härtig quantifier*

$$I_A XY \iff |X| = |Y|$$

is not the denotation of any English determiner. And EXT implies that no determiner can mean, say, *some* on universes with less than 10 elements and *all* on other universes. Nevertheless, the determiner denotations form a rich class of quantifiers: as further examples we may take the *proportional* quantifiers $|X \cap Y| > m/n \cdot |X|$ and $|X \cap Y| \geq m/n \cdot |X|$, $1 \leq m < n$, expressed by *more than (at least) m n:ths of the*, and Boolean combinations of these, such as *less than half of the, between ten and twenty percent of the, not more than ten or at least twenty percent of the*, etc.

Together, CONS and EXT express a characteristic asymmetry between the two arguments of determiner denotations: the role of the noun argument is to *restrict the domain of quantification.* A precise statement is the following

1.1 Fact. *A type $\langle 1,1 \rangle$ quantifier is CONS and EXT iff it is the relativization of some simple unary quantifier.*

Clearly, Q^{rel} is always CONS and EXT and, conversely, if Q' is CONS and EXT then, for Q defined by $Q_A Y \Leftrightarrow Q'_A AY$, $Q^{\text{rel}} = Q'$. This simple fact often enables one to generalize results about simple unary quantifiers to determiner denotations.

Notice also that a simple unary Q is monotone iff Q^{rel} is *right monotone* (monotone in the verb argument). For example, the proportional quantifiers above are right monotone.

Thus, NP semantics involves monadic quantifiers. But *sentences* can combine several NPs, together with transitive or ditransitive verbs denoting not sets but binary or ternary relations. Their truth conditions can then be given by means of polyadic quantifiers. Indeed, common sentential structures correspond to ways of *lifting* monadic quantifiers to polyadic ones. The most ubiquitous lift is *iteration*, as in

(3) Most students criticized three teachers

with a quantified subject and object ($\textit{most}\, X$ and $\textit{three}\, Y$) and a transitive verb (R). The truth conditions (of one of the two readings of (3)) are

(4) $\textit{most}\, x, y(Xx, \textit{three}\, z, u(Yz, Ryu))$

or, equivalently, the result of applying the lifted quantifier $\text{It}(\textit{most}, \textit{three})$ to the arguments X, Y and R: define (suppressing A)

(5) $\text{It}(Q_1, Q_2)XYR \iff Q_1 X\{a \mid Q_2 Y R_a\}$

($R_a = \{b \mid Rab\}$). However, seemingly similar sentences sometimes correspond to other lifts than iteration. For example,

(6) Forty researchers wrote thirty-two papers for the Handbook

presumably means neither that each of the forty researchers wrote thirty-two papers, nor that each of the thirty-two papers was written by forty (co)authors. Instead, this is an instance of so-called *cumulative* quantification: each of the forty researchers authored at least one paper for the Handbook, and each of the thirty-two papers was authored by at least one researcher. Thus, we have the lift

(7) $\mathrm{Cum}(Q_1, Q_2)XYR \iff \mathrm{It}(Q_1, \mathit{some})XYR \wedge \mathrm{It}(Q_2, \mathit{some})YXR^{-1}$.

We now come to the lifts studied in this paper. The first was introduced by Barwise in [1] with examples such as

(8) Most boys in your class and most girls in my class have all dated each other

Here the lift in question, on at least one plausible reading, is *branching*.

1.2 Definition. For right monotone type $\langle 1, 1\rangle$ quantifiers $Q_1, \ldots, Q_k$, define (again suppressing the universe)

$$\mathrm{Br}(Q_1, \ldots, Q_k)X_1 \ldots X_kR \iff \exists Y_1 \subseteq X_1 \ldots \exists Y_k \subseteq X_k[Q_1X_1Y_1 \wedge \ldots \wedge Q_kX_kY_k \wedge Y_1 \times \ldots \times Y_k \subseteq R].$$

We write $\mathrm{Br}^k(Q)$ for $\mathrm{Br}(Q, \ldots, Q)$ (k arguments). The branching of k monotone type $\langle 1\rangle$ quantifiers is defined analogously by suppressing $X_1, \ldots, X_k$.

So (8) means that there is a set X containing more than half the boys in your class and a set Y containing more than half the girls in my class such that any pair in $X \times Y$ is a 'dating' pair.

A similar lift occurs in certain reciprocal sentences like

(9) At least two thirds of the boys in your class all like each other

which can be taken as saying that there is a set containing at least two thirds of the boys in your class such that any two distinct boys in this set are in the 'like' relation:

1.3 Definition. Let Q be a right monotone type $\langle 1,1\rangle$ quantifier.

$$\mathrm{Ram}^k(Q)XR \iff \exists Y \subseteq X[QXY \ \wedge\ Y^k - I(Y) \subseteq R]$$

(where $I(Y) = \{(a_1,\ldots,a_k) \in Y^k \mid \exists i,j(i \neq j \wedge a_i = a_j)\}$). Similarly for a monotone simple unary Q.

We call the values of this lift *Ramsey quantifiers*, extending slightly the common usage of this term (where Q is the quantifier 'there exists a least $\aleph_\alpha$').

Our final example of a polyadic lift, *resumption*, amounts to using a monadic quantifier to quantify over k-tuples of individuals. This lift has found uses in computer science (cf. [4] and [16]) but can be given a linguistic motivation as well; cf. cases like

(10) Most neighbours like each other

(11) Most twins never separate

1.4 Definition. Let Q be a unary quantifier. Define

$$(\mathrm{Res}^k(Q))_A R_1 \ldots R_n \iff Q_{A^k} R_1 \ldots R_n$$

where n is the length of the type of Q.

The route to the lifts here went via linguistics, but they are natural in other contexts too. For example, to say that a graph contains so and so many vertices of degree 3 one iterates monadic quantifiers. But to say that a graph contains a clique of such and such size, one needs a Ramsey quantifier. Of course, one could also use second-order logic. But the Ramsey quantifier gives a better estimate of just how much expressive power one needs to add to first-order logic in this case.

To know how powerful the lifts are one needs first of all to answer the basic logical definability question:

When is Lift($Q_1,\ldots,Q_n$) definable in $\mathcal{L}(Q_1,\ldots,Q_n)$?

(The question is also significant in a linguistic context. For example, non-definability may imply that a certain kind of grammar for the corresponding expressions does not exist.) Trivially, iterations and Boolean combinations thereof are so definable. We shall prove below that the other polyadic lifts – branching, Ramseyfication and resumption – are usually not definable in this way.

2 Monotone quantifiers

From now on, unless otherwise noted, we restrict attention to finite models. Then, a monotone simple unary quantifier can be identified with a function $f : \omega \to \omega$ such that for all $n \in \omega$, $f(n) \leq n+1$; the quantifier Q_f corresponding to f says of a set X that it has at least $f(n)$ elements, where n is the cardinality of the universe (if $f(n) = n+1$ this is trivially false). More precisely, given f, define

$$(Q_f)_A X \iff |X| \geq f(|A|)$$

for $X \subseteq A$. Conversely, if Q is monotone, and

$$f(n) = \begin{cases} \text{least } k \text{ s. t. } \exists A \exists X \subseteq A(|A| = n \wedge |X| = k \wedge Q_A X) & \text{if such a } k \text{ exists} \\ n+1 & \text{otherwise} \end{cases}$$

then $Q = Q_f$.

Rather perspicuous definability criteria can be given for monotone simple unary quantifiers. A full characterization can be found in [19]; here we quote for illustration the following special cases:

2.1 Theorem.

1. ([11], [21]) *The quantifier Q_f is first order definable iff either $f(n)$ or $n - f(n)$ is eventually constant.*
2. ([19]) *Suppose that $\lim_{n\to\infty} f(n) = \lim_{n\to\infty}(n - f(n)) = \infty$ and that either $\lim_{n\to\infty}(n/2 - f(n)) = \infty$ or $\lim_{n\to\infty}(f(n) - n/2) = \infty$. Then Q_g is definable in $\mathcal{L}_{\omega\omega}(Q_f)$ iff at least one of $g(n), n - g(n), f(n) - g(n), n - f(n) - g(n)$ is eventually constant.*

In the second of these statements Q_f is assumed to satisfy a strong unboundedness condition. For most of this paper, the following weaker notion is sufficient.

2.2 Definition. We say that the function f (or quantifier Q_f) is *unbounded* if $\forall m \exists n (m \leq f(n) \leq n - m)$. Otherwise f is *bounded.*

Canonical examples of unbounded functions are $[\frac{n}{2}], [\sqrt{n}]$, and $[\log(n)]$ (where $[a]$ is the integer part of a). Typical bounded functions are $f(n) = 1$ (corresponding to $\exists$), $f(n) = n$ ($\forall$), and $f(n) = 1$ for even n and $f(n) = n - 1$ otherwise.

2.3 Theorem. *If f is bounded, then for any $k \geq 2$, $\mathrm{Br}^k(Q_f)$ and $\mathrm{Ram}^k(Q_f)$ are definable in $\mathcal{L}_{\omega\omega}(Q_f)$.*

Proof. By hypothesis, there is a number m such that for every n, either $f(n) = 0$, or $f(n) = n + 1$, or $0 < f(n) < m$, or $n - m < f(n) \leq n$. We can use Q_f to uniquely characterize each of these possibilities for $f(n)$. To see this, note first that $f(|A|) = 0 \iff A \models \xi$ and $f(|A|) = |A| + 1 \iff A \models \theta$, where ξ and θ are the sentences $Q_f x(x \neq x)$ and $\neg Q_f x(x = x)$, respectively. Second, for $p > 0$ we have $f(|A|) = p \iff A \models \varphi_p$, where φ_p is

$$\exists x_1 \ldots \exists x_p [\bigwedge_{1 \leq i < j \leq p} x_i \neq x_j \ \wedge \ Q_f y (\bigvee_{i=1}^{p} y = x_i) \ \wedge \ \neg Q_f y (\bigvee_{i=1}^{p-1} y = x_i)].$$

Third, let ψ_q be the conjunction of

$$\exists x_1 \ldots \exists x_q \exists x_{q+1} [\bigwedge_{1 \leq i < j \leq q+1} x_i \neq x_j \ \wedge \ Q_f y (\bigwedge_{i=1}^{q} y \neq x_i) \ \wedge \ \neg Q_f y (\bigwedge_{i=1}^{q+1} y \neq x_i)]$$

and $\neg \xi$. Then $f(|A|) = |A| - q > 0 \iff A \models \psi_q$.

Moreover, for each one of these finitely many possibilities, the branching condition

$$\exists X_1 \ldots \exists X_k [|X_1|, \ldots, |X_k| \geq f(|A|) \ \wedge \ X_1 \times \ldots \times X_k \subseteq R]$$

is expressible in first-order logic. Indeed, if $f(|A|) = p > 0$, this condition is equivalent to the sentence

$$\eta_p = \exists \mathbf{x}^1 \ldots \exists \mathbf{x}^k [\bigwedge_{1 \leq i \leq k} \alpha(\mathbf{x}^i) \ \wedge \ \bigwedge_{h \in H} R(x^1_{h(1)}, \ldots, x^k_{h(k)})],$$

where $\mathbf{x}^i = (x^i_1, \ldots, x^i_p)$, $1 \leq i \leq k$, $\alpha(\mathbf{x})$ states that all components of $\mathbf{x}$ are distinct, and H is the set of all functions $h : \{1, \ldots, k\} \to \{1, \ldots, p\}$. In the case $f(|A|) = |A| - q > 0,\ q > 0$, the branching condition is equivalent to the sentence

$$\zeta_q = \exists \mathbf{x}^1 \ldots \exists \mathbf{x}^k \forall \mathbf{y}[(\bigwedge_{1 \le i \le k} y_i \notin \mathbf{x}^i) \to R(y_1, \ldots, y_k)],$$

where $\mathbf{x}^i = (x_1^i, \ldots, x_q^i),\ 1 \le i \le k$, and $\mathbf{y} = (y_1, \ldots, y_k)$ (for $q = 0$, ζ_q is simply $\forall x_1, \ldots, x_k R(x_1, \ldots, x_k)$). And finally, if $f(|A|) = 0$ ($f(|A|) = |A| + 1$), then the branching condition is trivially true (false).

Putting all this together we conclude that $\mathrm{Br}^k(Q_f)x_1, \ldots, x_k R(x_1, \ldots, x_k)$ is equivalent to the sentence

$$\xi \ \vee \ \bigvee_{1 \le p < m} (\varphi_p \wedge \eta_p) \ \vee \ \bigvee_{q < m} (\psi_q \wedge \zeta_q).$$

The case of $\mathrm{Ram}^k(Q_f)$ is similar. □

The relativization Q_f^{rel} of Q_f is the right monotone type $\langle 1, 1 \rangle$ quantifier

$$(Q_f^{\mathrm{rel}})_A XY \iff |X \cap Y| \ge f(|X|).$$

We say that Q_f^{rel} is *bounded* if f is bounded. Notice that Q_f is definable in $\mathcal{L}_{\omega\omega}(Q_f^{\mathrm{rel}})$, that $\mathrm{Br}(Q_{f_1}, \ldots, Q_{f_k})$ is definable in $\mathcal{L}_{\omega\omega}(\mathrm{Br}(Q_{f_1}^{\mathrm{rel}}, \ldots, Q_{f_k}^{\mathrm{rel}}))$, and that $\mathrm{Ram}^k(Q_f)$ is definable in $\mathcal{L}_{\omega\omega}(\mathrm{Ram}^k(Q_f^{\mathrm{rel}}))$. From Fact 1.1 we see that right monotone determiner denotations usually are of the form Q_f^{rel}. For example, the quantifier *most* is Q_f^{rel} for $f(n) = [\frac{n}{2}] + 1$. And in most of these cases, relativization increases expressive power. In fact, the following result tells us precisely when this happens.

2.4 Theorem. ([11], [21]) *The quantifier Q_f^{rel} is definable in $\mathcal{L}_{\omega\omega}(Q_f)$ (or in $\mathcal{L}^{\omega}_{\infty\omega}(Q_f)$) iff either $f(n)$ or $n - f(n)$ is eventually constant (i.e., iff Q_f is first-order definable).*

Note that Q^{rel} is definable in $\mathcal{L}(Q)$ iff $\mathcal{L}(Q)$ has the *relativization property*, i.e., for any $\mathcal{L}(Q)$-sentence φ, $\varphi^{(P)}$ is equivalent to an $\mathcal{L}(Q)$-sentence. We end this section by stating a relativized version of Theorem 2.3 (these theorems will be generalized to the branching of several quantifiers in section 3).

2.5 Theorem. *If f is bounded, then* $\mathrm{Br}^k(Q_f^{rel})$ *and* $\mathrm{Ram}^k(Q_f^{rel})$ *are definable in* $\mathcal{L}_{\omega\omega}(Q_f^{rel})$.

Proof. Similar to the proof of Theorem 2.3. To formulate the branching condition $\mathrm{Br}^k(Q_f^{\mathrm{rel}})X_1 \dots X_k R$ over a universe A, we need to express statements of the form

$$f(|X_i|) = 0$$
$$f(|X_i|) = |X_i| + 1$$
$$f(|X_i|) = p$$
$$f(|X_i|) = |X_i| - q$$

($p > 0,\ q \geq 0$), and this can be done with the quantifier Q_f^{rel}. □

3 Definability of branching

In this section we first recall definitions and basic properties of some Ehrenfeucht-Fraïssé type games, and then apply these games to the definability of branching.

The *bijective Ehrenfeucht-Fraïssé game of length* q, $\mathrm{BEF}^q(\mathbf{A}, \mathbf{B})$, has two players, which we call *Duplicator* and *Spoiler*, respectively. In each round $1 \leq i \leq q$ of the game Duplicator chooses first a bijection $f_i : A \to B$ and Spoiler responds by choosing an element $a_i \in A$. These q pairs of moves of the players determine a relation $p = \{ (a_1, f_1(a_1)), \dots, (a_q, f_q(a_q)) \} \subseteq A \times B$. Duplicator wins the game if p is a *partial isomorphism* $\mathbf{A} \to \mathbf{B}$, i.e., if p is an injective function such that for each relation R of $\mathbf{A}$ and the corresponding relation R' of $\mathbf{B}$, and for all tuples $(b_1, \dots, b_k) \in (\mathrm{dom}(p))^k$,

$$(b_1, \dots, b_k) \in R \iff (p(b_1), \dots, p(b_k)) \in R'.$$

Spoiler wins if this is not the case, or if Duplicator cannot make his moves, i.e., if there are no bijections $A \to B$.

As usual, what happens in some particular play of $\mathrm{BEF}^q(\mathbf{A}, \mathbf{B})$ is not important. It is rather the question "which one of the players has a winning strategy" that is relevant for us. Here we say that Duplicator (Spoiler) has a *winning strategy* if he has a systematic way of choosing the bijections f_i

(the elements a_i) such that using it he always wins the game, no matter how the other player moves.

The following result is a direct consequence of Theorem 2.5 of [6]. For the sake of completeness we will explain here the basic idea behind its proof. Recall that $\mathbf{A} \equiv^q_{\mathcal{L}_{\omega\omega}(\mathbf{Q})} \mathbf{B}$ means that $\mathbf{A}$ and $\mathbf{B}$ satisfy the same $\mathcal{L}_{\omega\omega}(\mathbf{Q})$-sentences of quantifier rank at most q.

3.1 Proposition. *If Duplicator has a winning strategy in the game* $\mathrm{BEF}^q(\mathbf{A},\mathbf{B})$*, then* $\mathbf{A} \equiv^q_{\mathcal{L}_{\omega\omega}(\mathbf{Q}_1)} \mathbf{B}$.

Proof. (idea) Consider a formula $\varphi(\mathbf{y}) = Qx_1,\ldots,x_m\,(\psi_1(\mathbf{y},x_1),\ldots,\psi_m(\mathbf{y},x_m))$ of $\mathcal{L}_{\omega\omega}(\mathbf{Q}_1)$ and a partial function $p : A \to B$, $\mathbf{a} \mapsto \mathbf{b}$. If f is a bijection $A \to B$ such that for every $a \in A$ and each $1 \le j \le m$, $\mathbf{A} \models \psi_j(\mathbf{a},a) \Leftrightarrow \mathbf{B} \models \psi_j(\mathbf{b},f(a))$, then the function f is an isomorphism between the defined structures $(A,\psi_1^{\mathbf{A}}(\mathbf{a},\cdot),\ldots,\psi_m^{\mathbf{A}}(\mathbf{a},\cdot))$ and $(B,\psi_1^{\mathbf{B}}(\mathbf{b},\cdot),\ldots,\psi_m^{\mathbf{B}}(\mathbf{b},\cdot))$. Hence, $\mathbf{A} \models \varphi(\mathbf{a})$ if and only if $\mathbf{B} \models \varphi(\mathbf{b})$, no matter what the interpretation of the quantifier Q is. Using this observation it is easy to prove by induction on r that if $f_1,\ldots f_{q-r}$ are bijections $A \to B$ which Duplicator has chosen according to his winning strategy, and $a_1,\ldots,a_{q-r} \in A$ are the elements chosen by Spoiler, then

$$\mathbf{A} \models \varphi(a_1,\ldots,a_{q-r}) \iff \mathbf{B} \models \varphi(f_1(a_1),\ldots,f_{q-r}(a_{q-r}))$$

for every formula φ of $\mathcal{L}_{\omega\omega}(\mathbf{Q}_1)$ with $qr(\varphi) \le r$. In particular, the equivalence $\mathbf{A} \models \varphi \Leftrightarrow \mathbf{B} \models \varphi$ holds for every sentence φ of quantifier rank $\le q$. $\square$

If the structures $\mathbf{A}$ and $\mathbf{B}$ are finite, then the converse of Proposition 3.1 is also true: $\mathbf{A} \equiv^q_{\mathcal{L}_{\omega\omega}(\mathbf{Q}_1)} \mathbf{B}$ if and only if Duplicator has a winnning strategy in $\mathrm{BEF}^q(\mathbf{A},\mathbf{B})$. Furthermore, this equivalence holds for infinite structures, too, if the logic $\mathcal{L}_{\omega\omega}(\mathbf{Q}_1)$ is replaced with the infinitary logic $\mathcal{L}_{\infty\omega}(\mathbf{Q}_1)$ (cf. [6]). However, the implication stated in Proposition 3.1 is all that is needed for proving undefinability results.

3.2 Corollary. *Let* $\mathcal{K}$ *be a class of structures. If for every natural number* q *there exist structures* $\mathbf{A}$ *and* $\mathbf{B}$ *such that* $\mathbf{A} \in \mathcal{K}$, $\mathbf{B} \notin \mathcal{K}$ *and Duplicator has a winning strategy in* $\mathrm{BEF}^q(\mathbf{A},\mathbf{B})$*, then* $\mathcal{K}$ *is not definable in* $\mathcal{L}_{\omega\omega}(\mathbf{Q}_1)$.

Below we will give a non-trivial application of $\mathrm{BEF}^q(\mathbf{A},\mathbf{B})$ to finite models. But to get a feeling for the game, it may be helpful to to first see the use of Corollary 3.2 with a much simpler example, which involves infinite models.

3.3 Example. For each natural number m, let E_m be an equivalence relation on ω with m equivalence classes, all of cardinality ω. Similarly, let $E_\omega \subseteq \omega^2$ be an equivalence relation with ω equivalence classes of cardinality ω. We claim that $(\omega, E_m) \equiv^q_{\mathcal{L}_{\omega\omega}(\mathbf{Q}_1)} (\omega, E_n)$ whenever $q \leq m, n \leq \omega$. To see this, consider an arbitrary partial isomorphism $p = \{(a_1, b_1), \ldots, (a_r, b_r)\}$ such that $|p| = r < q$. For each i, let $A_i = \{\, a \in \omega - \{a_1, \ldots, a_r\} \mid (a, a_i) \in E_m \,\}$ and $B_i = \{\, b \in \omega - \{b_1, \ldots, b_r\} \mid (b, b_i) \in E_n \,\}$. Clearly all these sets $A_1, \ldots, A_r$ and $B_1, \ldots, B_r$ are of cardinality ω. Furthermore, since $r < q \leq m, n$, the sets $A = \omega - (A_1 \cup \cdots \cup A_r)$ and $B = \omega - (B_1 \cup \cdots \cup B_r)$ are also of cardinality ω. Hence, there exists a bijection $f : \omega \to \omega$ extending p which maps A_i onto B_i for $1 \leq i \leq r$, and A onto B. But then $p \cup \{(a, f(a))\}$ is a partial isomorphism $(\omega, E_m) \to (\omega, E_n)$ for any $a \in \omega$. In particular, Duplicator can keep choosing bijections f_i, $1 \leq i \leq q$, in the game $\mathrm{BEF}^q((\omega, E_m), (\omega, E_n))$ in such a way that the mapping $\{\, (a_1, f_1(a_1)), \ldots, (a_i, f_i(a_i)) \,\}$ is always a partial isomorphism.

Let $\mathcal{K}$ be the class of all structures (A, E) such that E is an equivalence relation on A with finitely many equivalence classes. Since $(\omega, E_m) \in \mathcal{K}$ for every $m < \omega$, but $(\omega, E_\omega) \notin \mathcal{K}$, it follows from Corollary 3.2 that $\mathcal{K}$ is not definable in $\mathcal{L}_{\omega\omega}(\mathbf{Q}_1)$. In a similar fashion we see that the class $\mathcal{K}'$ consisting of those equivalence relations (A, E) that have an even number of equivalence classes is not definable in $\mathcal{L}_{\omega\omega}(\mathbf{Q}_1)$.

Historical remark. The example is a variant of Keisler's counterexample to interpolation in $\mathcal{L}_{\omega\omega}(Q_1)$. It was realized by Caicedo [3] and Väänänen (cf. [12]) that this kind of example (though with infinitely many equivalence classes) works for all monadic quantifiers; a proof using the BEF game was given in [6].

3.1 Branching of one quantifier

In Section 2 we saw that branching a bounded quantifier does not have any effect on its expressive power. However, the following result shows that branching usually increases the expressive power even beyond the reach of any monadic quantifiers.

3.4 Theorem. *Let Q be a monotone simple monadic quantifier. If Q is unbounded, then* $\mathrm{Br}^2(Q)$ *is not definable in* $\mathcal{L}_{\omega\omega}(\mathbf{Q}_1)$.

Putting together Theorems 2.3 and 3.4 we obtain a complete characterization of the definability of $\mathrm{Br}^2(Q)$ in terms of Q for any monotone simple monadic quantifier Q. Using Theorem 2.5 (and the fact that $\mathrm{Br}^2(Q_f)$ is definable in $\mathcal{L}_{\omega\omega}(\mathrm{Br}^2(Q_f^{\mathrm{rel}}))$) we get the same characterization for relativizations of such quantifiers. Moreover, 2 can be replaced by any $k > 1$, since we have

3.5 Lemma. *If Q is a monotone simple unary quantifier, or the relativization of such a quantifier, then* $\mathrm{Br}^k(Q)$ *is definable in* $\mathcal{L}_{\omega\omega}(\mathrm{Br}^{k+1}(Q))$.

Proof. When $Q = Q_f$, we have simply

$$\mathrm{Br}^k(Q)x_1,\dots,x_k R(x_1,\dots,x_k) \leftrightarrow \mathrm{Br}^{k+1}(Q)x_1,\dots,x_{k+1} R(x_1,\dots,x_k)$$

which is seen by considering separately the three cases (1) $f(|A|) = |A| + 1$, (2) $f(|A|) = 0$, and (3) $f(|A|) \neq 0, |A| + 1$. The relativized case is similar. □

Applying also Fact 1.1, we obtain the following characterization.

3.6 Corollary. *Suppose that Q is either a monotone type $\langle 1 \rangle$ quantifier, or a right monotone type $\langle 1, 1 \rangle$ quantifier satisfying CONS and EXT. Then the following conditions are equivalent, for any $k > 1$:*

(1) $\mathrm{Br}^k(Q)$ *is definable in* $\mathcal{L}_{\omega\omega}(Q)$.

(2) $\mathrm{Br}^k(Q)$ *is definable in* $\mathcal{L}_{\omega\omega}(\mathbf{Q}_1)$.

(3) Q *is bounded.*

In [9], Hella and Sandu proved, modifying a model construction by Fagin [5], that connectivity of (finite) graphs is not definable in terms of monadic quantifiers. Their proof implies a special case of Corollary 3.6: $\mathrm{Br}^2(\mathit{most})$ is not definable in $\mathcal{L}_{\omega\omega}(\mathbf{Q}_1)$ (cf. [22]). Here we will use a simple trick to modify this proof so that it works for any unbounded quantifier Q_f.

3.7 Definition. Let n be a natural number and $\mathbf{A} = (A, E)$ a graph. We say that $\mathbf{A}$ is *n-separable* if there exist subsets $C, D \subseteq A$ such that $|C| = |D| = n$ and $(C \times D) \cap E = \emptyset$. Furthermore, if f is a function $\omega \to \omega$, we denote by $\mathcal{K}_f$ the class of all graphs $\mathbf{A}$ which are $f(|A|)$-separable.

Here all graphs are assumed to be undirected, but not necessarily irreflexive. Thus, a structure (A, E) is a graph if E is a symmetric binary relation on A, possibly containing self-loops (a, a).

Observe now that a graph (A, E) is in the class $\mathcal{K}_f$ if and only if its complement graph $(A, A^2 - E)$ is in the quantifier $\mathrm{Br}^2(Q_f)$. Hence, we have

3.8 Lemma. *For any function $f : \omega \to \omega$, the class $\mathcal{K}_f$ is definable in the logic $\mathcal{L}_{\omega\omega}(\mathrm{Br}^2(Q_f))$.*

We shall now define two families of graphs that are the heart of the proof of Theorem 3.4.

3.9 Definition. Let q, r and s be natural numbers, and let P, P', R and S be mutually disjoint sets of cardinalities $2^q{+}2$, $2^q{+}2$, r and s, respectively. Assume further that $P = \{c_0, \ldots, c_{2t-1}\}$ and $P' = \{d_0, \ldots, d_{t-1}, e_0, \ldots, e_{t-1}\}$, where $t = 2^{q-1}{+}1$. We define two graphs $\mathbf{A}_{q,r,s} = (A, E)$ and $\mathbf{B}_{q,r,s} = (B, E')$ as follows:

- $A = P \cup R \cup S$;
- $B = P' \cup R \cup S$;
- $E = (S \times A) \cup (A \times S) \cup \{\, (c_i, c_j) \mid |i - j| \in \{0, 1, 2t-1\} \,\}$;
- $E' = (S \times B) \cup (B \times S) \cup \{\, (d_i, d_j), (e_i, e_j) \mid |i - j| \in \{0, 1, t-1\} \,\}$.

In the case $r = s = 0$ we denote $\mathbf{A}_{q,r,s}$ and $\mathbf{B}_{q,r,s}$ simply by $\mathbf{A}_q$ and $\mathbf{B}_q$.

Thus, one can visualize the graphs $\mathbf{A}_{q,r,s}$ and $\mathbf{B}_{q,r,s}$ as follows. $\mathbf{A}_{q,r,s}$ consists of a cycle (with self-loops) of length $2^q + 2$ and two 'boxes' R and S containing r and s elements, respectively. Each element in S is connected by an edge to all elements of A (including itself), whereas the elements in R are not adjacent to any elements other than those in S. The graph $\mathbf{B}_{q,r,s}$ is similar, except that instead of the big cycle it contains two cycles of length $2^{q-1} + 1$.

3.10 Lemma. *For all natural numbers q, r and s, Duplicator has a winning strategy in the game $\mathrm{BEF}^q(\mathbf{A}_{q,r,s}, \mathbf{B}_{q,r,s})$.*

Proof. We start by observing that the subsets R and S of the universe of $\mathbf{A}_{q,r,s}$ and $\mathbf{B}_{q,r,s}$ can be omitted in this proof without loss of generality: Duplicator has a winning strategy in $\mathrm{BEF}^q(\mathbf{A}_{q,r,s}, \mathbf{B}_{q,r,s})$ if and only if he has one in $\mathrm{BEF}^q(\mathbf{A}_q, \mathbf{B}_q)$. Indeed, the mapping $f \mapsto f \cup \mathrm{id}_R \cup \mathrm{id}_S$ for each bijection $f : P \to P'$, transforms any winning strategy of Duplicator in $\mathrm{BEF}^q(\mathbf{A}_q, \mathbf{B}_q)$ to a winning strategy in $\mathrm{BEF}^q(\mathbf{A}_{q,r,s}, \mathbf{B}_{q,r,s})$. On the other hand, if Duplicator wins the game between the larger structures $\mathbf{A}_{q,r,s}$ and $\mathbf{B}_{q,r,s}$, then he wins the game between $\mathbf{A}_q$ and $\mathbf{B}_q$ simply by choosing the restrictions of his bijections to the set P. Hence, it suffices to show that Duplicator can always win the game $\mathrm{BEF}^q(\mathbf{A}_q, \mathbf{B}_q)$. This was already done in [9] (for cycles without self-loops), but for the sake of completeness we give here a new proof.

For each $k < 2t = 2^q + 2$, let $g_k : P \to P'$ be the bijection defined by

$$g_k(c_i) = \begin{cases} d_i, & \text{if } i \equiv j \pmod{2t} \text{ for some } j \in [k, t+k-1] \\ e_i, & \text{if } i \equiv j \pmod{2t} \text{ for some } j \in [t+k, 2t+k-1] \end{cases}$$

(Here we are stipulating that if $t \le i < 2t$, then $d_i = d_{i-t}$, and similarly for e_i.) Thus, g_k arises by splitting the big cycle in $\mathbf{A}_q$ into two halves and mapping each of these halves, in a uniform way, onto one of the small cycles in $\mathbf{B}_q$. We will show below that Duplicator wins the game $\mathrm{BEF}^q(\mathbf{A}_q, \mathbf{B}_q)$ by choosing bijections of the form g_k for suitable k.

Before describing this winning strategy we need to introduce some auxiliary concepts. Let C be a subset of P and k and l integers less than $2t$. We say that g_k and g_l are *C-equivalent* if $g_l(a) = g_k(a)$ for every $a \in C$. The *splitting points* of g_k are c_{k-1}, c_k, c_{t+k-1} and c_{t+k}. The *distance* of C from the splitting corresponding to g_k is

$$d_k(C) = \min\{\, d(a,b) \mid a \in C, b \text{ a splitting point of } g_k\},$$

where $d(a,b)$ denotes the usual distance between elements in a graph. Furthermore, for each $a \in P$, we denote by $h_k(a)$ the splitting point which is closest to a. Note that if $d_k(C) > 0$, then the restriction of the bijection g_k to the set $C \cup \{a\}$ is a partial isomorphism $\mathbf{A}_q \to \mathbf{B}_q$ for any $a \in P$.

We claim now that Duplicator can choose his bijections f_i, $1 \le i \le q$, in the game $\mathrm{BEF}^q(\mathbf{A}_q, \mathbf{B}_q)$ in such a way that, for each $1 \le i \le q$,

(1) there is $k < 2t$ such that $f_i = g_k$ and $d_k(\{a_1, \dots, a_{i-1}\}) \ge 2^{q-i}$, and

(2) f_i is $\{a_1, \dots, a_{i-1}\}$-equivalent to f_{i-1},

where $a_1, \ldots, a_q \in A$ are the elements chosen by Spoiler. This constitutes the promised winning strategy for Duplicator, since condition (2) implies that the mapping $\{ (a_1, f_1(a_1)), \ldots, (a_q, f_q(a_q)) \}$ is the restriction of f_q to the set $\{a_1, \ldots, a_q\}$, and this restriction is a partial isomorphism by condition (1) for $i = q$. The claim is proved by induction on i:

(i) We let $f_1 = g_0$, and note that conditions (1) and (2) are trivially satisfied ($d_0(\emptyset) = \infty$).

(ii) Let $a_1 = c_j$. If $j \le t-1$, then we let $f_2 = g_k$ where k is the unique natural number $< 2t$ such that $k + 2^{q-2} \equiv j \pmod{2t}$. Otherwise let $f_2 = g_k$ for the unique k such that $k - 2^{q-2} \equiv j \pmod{2t}$. In both cases $d_k(\{a_1\}) = 2^{q-2}$, and $f_2(a_1) = g_k(c_j) = g_0(c_j) = f_1(a_1)$.

(iii) Let $1 < i \le q$, and assume as induction hypothesis that conditions (1) and (2) hold for the bijection $f_i = g_k$. There are two possibilities: If $h_k(a_i) \in \{c_{k-1}, c_{t+k-1}\}$, we choose $f_{i+1} = g_l$ for $l \equiv k + 2^{q-i-1} \pmod{2t}$. Then we have

$$\begin{aligned} d_l(\{a_1, \ldots, a_i\}) &= \min\{d_l(\{a_1, \ldots, a_{i-1}\}), d_l(\{a_i\})\} \\ &\ge \min\{d_k(\{a_1, \ldots, a_{i-1}\}) - 2^{q-i-1}, d(c_{k-1}, c_{l-1})\} = 2^{q-i-1}, \end{aligned}$$

and since there are no elements of $\{a_1, \ldots, a_i\}$ between c_k and c_l or between c_{t+k} and c_{t+l}, g_l is $\{a_1, \ldots, a_i\}$-equivalent to g_k. In the case $h_k(a_i) \in \{c_k, c_{t+k}\}$ we let $f_{i+1} = g_l$ for $l \equiv k - 2^{q-i-1} \pmod{2t}$; conditions (1) and (2) are seen to hold by a similar argument.

□

Remark. A slightly weaker version of Lemma 3.10 can also be proved by using a recent result of Nurmonen [18]. This alternative argument goes as follows. For each natural number e, the *e-type* of an element a of a graph $\mathbf{A}$ is the isomorphism type of the structure $(\mathbf{N}_e(a), a)$, where $\mathbf{N}_e(a)$ is the subgraph of $\mathbf{A}$ generated by the set of all elements b such that $d(b, a) < e$. Two graphs $\mathbf{A}$ and $\mathbf{B}$ are *e-equivalent* if there is a bijection $f : A \to B$ which preserves e-types. Nurmonen proved that if $\mathbf{A}$ and $\mathbf{B}$ are 3^q-equivalent, then Duplicator has a winning strategy in $\mathrm{BEF}^q(\mathbf{A}, \mathbf{B})$. It is easy to see that $\mathbf{A}_{q',r,s}$ and $\mathbf{B}_{q',r,s}$ are always $2^{q'-2}$-equivalent, and consequently Duplicator has a winning strategy in $\mathrm{BEF}^q(\mathbf{A}_{q',r,s}, \mathbf{B}_{q',r,s})$ whenever $2^{q'-2} \ge 3^q$.

Proof of Theorem 3.4. Note first that in the graph $\mathbf{B}_{q,r,s}$ there are no edges between the sets $C = \{d_0, \ldots, d_{t-1}\} \cup R$ and $D = \{e_0, \ldots, e_{t-1}\} \cup R$, for $t = 2^{q-1}+1$, whence $\mathbf{B}_{q,r,s}$ is $(t+r)$-separable. On the other hand, $\mathbf{A}_{q,r,s}$ is not $(t+r)$-separable, since if $C, D \subseteq A$ are sets of cardinality $t+r$, then either there is an element a of S in C or D, or there are elements $c_i \in C$ and $c_j \in D$ such that $|i-j| \in \{0, 1, 2t-1\}$. In the former case, there is an edge between a and every element of C and D, and in the latter case, there is an edge between c_i and c_j (here is where the self-loops are used).

Assume then that $Q = Q_f$ is unbounded, and q is a natural number. Since f is unbounded, we can choose n such that $t \leq f(n) \leq n-t$ where $t = 2^{q-1}$. Set $r = f(n) - t$ and $s = n - t - f(n)$. Thus, n equals the size of the graphs $\mathbf{A}_{q,r,s}$ and $\mathbf{B}_{q,r,s}$, and $f(n) = t + r$. By the observation above, $\mathbf{B}_{q,r,s}$ is $f(n)$-separable, but $\mathbf{A}_{q,r,s}$ is not. From Corollary 3.2 and Lemma 3.10 it follows that the class $\mathcal{K}_f$ is not definable in $\mathcal{L}_{\omega\omega}(\mathbf{Q}_1)$. Since, by Lemma 3.8, $\mathcal{K}_f$ is definable in $\mathcal{L}_{\omega\omega}(\mathrm{Br}^2(Q_f))$, it follows that $\mathrm{Br}^2(Q_f)$ is not definable in $\mathcal{L}_{\omega\omega}(\mathbf{Q}_1)$. $\square$

3.2 Branching of several quantifiers

We shall now extend the characterization in Corollary 3.6 to the branching of more than one quantifier. The first thing to observe is that if both f and g are unbounded, it does not necessarily follow that $\mathrm{Br}(Q_f, Q_g)$ is undefinable in $\mathcal{L}_{\omega\omega}(Q_f, Q_g)$. For example, let $f(n)$ be $n/2$ if n is even and $n+1$ otherwise, and let $g(n)$ be $(n+1)/2$ if n is odd and $n+1$ otherwise. Then $\mathrm{Br}(Q_f, Q_g)$ is trivially false of every relation, hence definable in $\mathcal{L}_{\omega\omega}$, although $\mathrm{Br}^2(Q_f)$ and $\mathrm{Br}^2(Q_g)$ are both undefinable in $\mathcal{L}_{\omega\omega}(\mathbf{Q}_1)$.

What we need for $\mathrm{Br}(Q_f, Q_g)$ is that f and g are 'jointly unbounded' in the sense that $\forall t \exists n (t \leq f(n), g(n) \leq n - t)$. Then the proof of Theorem 3.4 can be modified to show that $\mathrm{Br}(Q_f, Q_g)$ is not definable in $\mathcal{L}_{\omega\omega}(\mathbf{Q}_1)$. And in fact the converse also holds: if f, g are not jointly unbounded it can be shown that $\mathrm{Br}(Q_f, Q_g)$ is definable in $\mathcal{L}_{\omega\omega}(Q_f, Q_g)$. (Both of these claims will follow from Theorem 3.12 below.)

But these observations on the branching of two quantifiers do not generalize immediately to the branching of k quantifiers. First, the condition on f, g, h of not being jointly unbounded in the above sense turns out to be too weak to permit the conclusion that $\mathrm{Br}(Q_f, Q_g, Q_h)$ is definable in $\mathcal{L}_{\omega\omega}(Q_f, Q_g, Q_h)$. And second, Lemma 3.5 does not generalize directly to the branching of several quantifiers. This is because on a model $\mathbf{A}$ where $h(|A|) = |A|+1$, $\mathrm{Br}(Q_f, Q_g, Q_h)xyz\,\varphi$ is always false, whereas if $h(|A|) = 0$, we have

$$\mathrm{Br}(Q_f, Q_g, Q_h)xyz\,\varphi \iff Q_f x(x = x) \wedge Q_g y(y = y).$$

If we know that $h(n)$ is never 0 or $n+1$, then $\mathrm{Br}(Q_f, Q_g)$ is definable in terms of $\mathrm{Br}(Q_f, Q_g, Q_h)$ as in the proof of Lemma 3.5, but not necessarily otherwise.

It turns out, however, that with a more careful generalization of the notion of (un)boundedness to sequences of functions, we can obtain a necessary and sufficient condition for the definability of the branching of k quantifiers.

3.11 Definition. Let $k > 1$. $\langle f_1, \dots, f_k\rangle$ is *bounded* if there exists a number t such that for all n, either $f_i(n) = n+1$ for some i, or $f_i(n) = 0$ for some i, or $t \leq f_i(n) \leq n-t$ holds for at most one i. Otherwise $\langle f_1, \dots, f_k\rangle$ is *unbounded*. We also say that $\langle Q_{f_1}, \dots, Q_{f_k}\rangle$ and $\langle Q^{\mathrm{rel}}_{f_1}, \dots, Q^{\mathrm{rel}}_{f_k}\rangle$ are (un)bounded under these circumstances.

The relation to the previous concept of boundedness for one function is the following:

If $f_1 = \ldots = f_k = f$, then f is bounded iff $\langle f_1, \dots, f_k\rangle$ is bounded

(observe that if P holds of f_i for at most one i, $k > 1$, and all the f_i are equal, then P does not hold of any f_i). Also, note that if each of $f_1, \dots, f_k$ is bounded, or even if all but one of them are bounded, then $\langle f_1, \dots, f_k\rangle$ is bounded.

The following theorem generalizes Corollary 3.6.

3.12 Theorem. *Let $\langle Q_1, \dots, Q_k\rangle$ be either a sequence of monotone type $\langle 1\rangle$ quantifiers, or a sequence of right monotone type $\langle 1,1\rangle$ quantifiers satisfying CONS and EXT ($k > 1$). Then the following conditions are equivalent:*

(1) $\mathrm{Br}(Q_1, \dots, Q_k)$ *is definable in* $\mathcal{L}_{\omega\omega}(Q_1, \dots, Q_k)$.

(2) $\mathrm{Br}(Q_1, \dots, Q_k)$ *is definable in* $\mathcal{L}_{\omega\omega}(\mathbf{Q}_1)$.

(3) $\langle Q_1, \dots, Q_k\rangle$ *is bounded.*

Proof. Let $\langle Q_1, \dots, Q_k\rangle = \langle Q_{f_1}, \dots, Q_{f_k}\rangle$.

(1) $\Rightarrow$ (2): Trivial.

(2) $\Rightarrow$ (3): Suppose $\langle f_1, \dots, f_k\rangle$ is unbounded.

Claim: There are i, j with $1 \leq i, j \leq k$ and $i \neq j$ such that for every t there is n such that

(i) for all l, $f_l(n) \neq 0$ and $f_l(n) \neq n+1$

(ii) $t \leq f_i(n), f_j(n) \leq n - t$.

For, by the unboundedness of $\langle f_1, \ldots, f_k \rangle$ it holds that for every t there is n and a pair i, j with $i \neq j$ such that (i) and (ii) hold. But there are only finitely many pairs i, j, so for at least one of them, (ii) holds for infinitely many t. From this the Claim readily follows.

Let, then, f_i and f_j be as in the Claim. Note that now we are essentially back to the case mentioned above with two 'jointly unbounded' functions. The proof of Theorem 3.4 is modified as follows. In the two models $\mathbf{A}_{q,r,s}$ and $\mathbf{B}_{q,r,s}$, split the 'box' R into two subsets R_1 and R_2 with r_1 and r_2 elements, respectively. R_1 and R_2 need not be disjoint but their union is R. Expand the models to $\mathbf{A}_{q,r_1,r_2,s}$ and $\mathbf{B}_{q,r_1,r_2,s}$ by adjoining two unary predicate symbols P_1 and P_2, which in $\mathbf{A}_{q,r_1,r_2,s}$ are interpreted as $P \cup R_1$ and $P \cup R_2$, respectively, and in $\mathbf{B}_{q,r_1,r_2,s}$ as $P' \cup R_1$ and $P' \cup R_2$. Since elements in R_1 and R_2 are only connected to elements in S, Duplicator still has a winning strategy in the game $\mathrm{BEF}^q(\mathbf{A}_{q,r_1,r_2,s}, \mathbf{B}_{q,r_1,r_2,s})$. Consider the sentence

$$\mathrm{Br}(Q_{f_1}, \ldots, Q_{f_k})x_1, \ldots, x_k\,(P_1(x_i) \wedge P_2(x_j) \wedge \neg E(x_i, x_j))$$

Take any quantifier depth q and choose n such that (i) and (ii) of the Claim hold. Say $f_i(n) \leq f_j(n)$. Let $r_1 = f_i(n) - t$, $r_2 = r = f_j(n) - t$, and $s = n - t - f_j(n)$. So we take $R_2 = R$ of size r_2, and R_1 as a subset of R of size r_1. Then n is the size of the models $\mathbf{A}_{q,r_1,r_2,s}$ and $\mathbf{B}_{q,r_1,r_2,s}$ and $f_i(n) = t + r_1$, $f_j(n) = t + r_2$. It is then easy to see, using the fact that none of $f_1(n), \ldots, f_k(n)$ is 0 or $n+1$, that the sentence above is true in $\mathbf{B}_{q,r_1,r_2,s}$ but false in $\mathbf{A}_{q,r_1,r_2,s}$.

(3) $\Rightarrow$ (1): Assume that $\langle f_1, \ldots, f_k \rangle$ is bounded and let t be the bound. For $1 \leq i \leq k$ and $r \leq t$, let θ_i, ξ_i, φ_i^r and ψ_i^r be $\mathcal{L}_{\omega\omega}(Q_{f_i})$-sentences such that (cf. the proof of Theorem 2.3) in a model with universe A,

$$\begin{aligned} \theta_i &\text{ says that } f_i(|A|) = |A| + 1 \\ \xi_i &\text{ says that } f_i(|A|) = 0 \\ \varphi_i^r &\text{ says that } f_i(|A|) = r \\ \psi_i^r &\text{ says that } f_i(|A|) = |A| - r \end{aligned}$$

Let $\Theta = \theta_1 \vee \ldots \vee \theta_k$ and $\Xi = \xi_1 \vee \ldots \vee \xi_k$. Finally, let $\Phi_1, \ldots, \Phi_s$ be a list of all conjunctions built up from the sentences $\neg\theta_i$, $\neg\xi_i$, φ_i^r and ψ_i^r which specify the value of $f_i(|A|)$ for all but one i, and also state that $f_i(|A|)$ is not $|A|+1$ or 0 for any i.

It follows from the assumption that in any model, either Θ or Ξ or one of $\Phi_1, \ldots, \Phi_s$ is true. In models of Θ the branching sentence

$$\mathrm{Br}(Q_{f_1}, \ldots, Q_{f_k})x_1, \ldots, x_k R(x_1, \ldots, x_k)$$

is false, and in models of $\neg\Theta \wedge \Xi$ it is true. Moreover, we make the following

Claim: For each Φ_j there is a sentence Ψ_j in $\mathcal{L}_{\omega\omega}(Q_{f_1}, \ldots, Q_{f_k})$ such that in models of Φ_j,

$$\mathrm{Br}(Q_{f_1}, \ldots, Q_{f_k})x_1, \ldots, x_k R(x_1, \ldots, x_k) \ \leftrightarrow \ \Psi_j .$$

¿From this it follows that the branching sentence is logically equivalent to the sentence

$$(\neg\Theta \wedge \Xi) \ \vee \ \bigvee_{j=1}^{s}(\Phi_j \wedge \Psi_j).$$

Proof of the Claim: An example will suffice to give the idea. Suppose that $k = 3$ and that Φ_j says that $f_2(|A|) = p$, $f_3(|A|) = |A| - q$, and that none of $f_1(|A|)$, $f_2(|A|)$, $f_3(|A|)$ is $|A|+1$ or 0. Then Ψ_j (written semi-formally) is

$$\exists \text{ distinct } y_1, \ldots, y_p \ \exists \text{ distinct } z_1, \ldots, z_q$$
$$Q_{f_1}x_1 \forall x_2 \forall x_3 (x_2 \in \{y_1, \ldots, y_p\} \wedge x_3 \notin \{z_1, \ldots, z_q\} \ \rightarrow \ R(x_1, x_2, x_3)).$$

(We have $p > 0$ and $|A| - q > 0$. If $q = 0$, delete "$\exists$ distinct $z_1, \ldots, z_q$" and the conjunct "$x_3 \notin \{z_1, \ldots, z_q\}$" above.) For this sentence says that there are sets Y and Z ($Z = A - \{z_1, \ldots, z_q\}$) such that $|Y| = p$ and $|Z| = |A| - q$ and

$$(A, \{a_1 \mid \forall a_2, a_3 (a_2 \in Y \wedge a_3 \in Z \ \rightarrow \ R(a_1, a_2, a_3))\}) \in Q_{f_1}.$$

Using the monotonicity of Q_{f_1} one sees that this is the same as saying that there are sets X, Y, Z such that $|Y| = p$ and $|Z| = |A| - q$ and

$$(A, X) \in Q_{f_1} \ \wedge \ X \times Y \times Z \subseteq R$$

which, by the monotonicity of Q_{f_2} and Q_{f_3}, is precisely what the branching sentence $\mathrm{Br}(Q_{f_1}, Q_{f_2}, Q_{f_3})x_1, x_2, x_3 R(x_1, x_2, x_3)$ says, provided Φ_j holds.

This proves the Claim, and thereby the Theorem in the unrelativized case. The case with relativized quantifiers is similar (cf. the proof of Theorem 2.5). □

4 Definability of Ramseyfication

In this section we show that the Ramsey lift $\mathrm{Ram}^{k+1}(Q)$ of an unbounded quantifier Q is not definable in $\mathcal{L}_{\omega\omega}(Q)$. So the situation is analogous to the case of the branching lift: the lift of Q is definable in $\mathcal{L}_{\omega\omega}(Q)$ if and only if Q is bounded. However, in two important respects our results about the Ramsey lift are stronger. Namely, we prove that for unbounded Q the lift $\mathrm{Ram}^{k+1}(Q)$ is not definable even in $\mathcal{L}^{\omega}_{\infty\omega}(\mathbf{Q}_k)$. Recall that $\mathcal{L}^{\omega}_{\infty\omega}$ is the fragment of the infinitary language $\mathcal{L}_{\infty\omega}$ in which every formula contains finitely many different variables only, and $\mathbf{Q}_k$ is the family of all k-ary generalized quantifiers. Thus we can prove the undefinability of $\mathrm{Ram}^{k+1}(Q)$ even with respect to infinite disjunctions and conjunctions and even with arbitrary k-ary generalized quantifiers.

The main interest of $\mathcal{L}^{\omega}_{\infty\omega}(\mathbf{Q}_k)$ is in the fact that it contains various *fixpoint extensions* of $\mathcal{L}_{\omega\omega}(\mathbf{Q}_k)$ (see [7, 11]). Intuitively speaking, $\mathcal{L}^{\omega}_{\infty\omega}(\mathbf{Q}_k)$ extends $\mathcal{L}_{\omega\omega}(\mathbf{Q}_k)$ by allowing (among other things) recursive definitions.

In Section 3 we introduced the game $\mathrm{BEF}^q(\mathbf{A},\mathbf{B})$. The point of this was that if Duplicator has a winning strategy in $\mathrm{BEF}^q(\mathbf{A},\mathbf{B})$, then $\mathbf{A} \equiv^q_{\mathcal{L}_{\omega\omega}(\mathbf{Q}_1)} \mathbf{B}$. We shall now recall a modification $\mathrm{BP}^l_k(\mathbf{A},\mathbf{B})$ of $\mathrm{BEF}^q(\mathbf{A},\mathbf{B})$ from [7] in order to get a criterion for $\mathbf{A} \equiv_{\mathcal{L}^l_{\infty\omega}(\mathbf{Q}_k)} \mathbf{B}$.

The *k-bijective l-pebble Ehrenfeucht-Fraïssé game*, $\mathrm{BP}^l_k(\mathbf{A},\mathbf{B})$ is defined as follows: In each round q of the game Duplicator chooses first a bijection $f_q : A \to B$ and Spoiler responds by choosing sets $C_q \subseteq A$ and $D_q \subseteq A$ so that $|D_q| \leq k$ and $|C_q \cup D_q| \leq l$. Intuitively speaking, Duplicator claims that f_q is an isomorphism and Spoiler tries to dispute this by pointing to the part $C_q \cup D_q$ of $\mathbf{A}$ where he thinks f_q does not preserve structure. To check this out the referee defines $p_q = (p_{q-1} \restriction C_q) \cup (f_q \restriction D_q)$ (letting $p_0 = \emptyset$). An extra rule dictates that Spoiler has to play in such a way that $C_q \subseteq dom(p_{q-1})$. Duplicator wins the game if for all q the relation p_q is a partial isomorphism $\mathbf{A} \to \mathbf{B}$. Spoiler wins if this is not the case, or if Duplicator cannot make his moves, i.e., if there are no bijections $A \to B$.

The difference between $\mathrm{BP}^l_1(\mathbf{A},\mathbf{B})$ and $\mathrm{BEF}^q(\mathbf{A},\mathbf{B})$ is the following: In the latter game all moves of Spoiler remain in the game and Duplicator has to create bigger and bigger partial isomorphisms, whereas in the former game Spoiler can keep only up to l previously played elements so the task of Duplicator is easier but, on the other hand, Duplicator does not know how many moves the game has. The game $\mathrm{BP}^l_k(\mathbf{A},\mathbf{B})$ introduces the new feature that Spoiler can let the set D_q of "new" elements contain up to k elements.

4.1 Proposition. $\mathbf{A} \equiv_{\mathcal{L}^l_{\infty\omega}(\mathbf{Q}_k)} \mathbf{B}$ *iff Duplicator has a winning strategy in the game* $\mathrm{BP}^l_k(\mathbf{A},\mathbf{B})$.

Proof. See the proof of Proposition 3.1 and the proof of Corollary 5.9 in [7]. □

4.2 Corollary. *Let Q be a quantifier. If for some sentence φ of $\mathcal{L}_{\omega\omega}(Q)$ and for all natural numbers l there exist structures $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{A} \models \varphi$, $\mathbf{B} \not\models \varphi$ and Duplicator has a winning strategy in* $\mathrm{BP}^l_k(\mathbf{A},\mathbf{B})$, *then Q is not definable in $\mathcal{L}^\omega_{\infty\omega}(\mathbf{Q}_k)$.*

We shall consider the lift $\mathrm{Ram}^2(Q)$ first, because the construction behind our result on the general case of $\mathrm{Ram}^k(Q)$ is different.

4.3 Theorem. *If f is unbounded, then* $\mathrm{Ram}^2(Q_f)$ *is not definable in $\mathcal{L}^\omega_{\infty\omega}(\mathbf{Q}_1)$.*

Proof. Let $l \geq 1$ be arbitrary. We show that $\mathrm{Ram}^2(Q_f)$ is not definable in $\mathcal{L}^l_{\infty\omega}(\mathbf{Q}_1)$. The proof is based on a modification of the proof of the main result of [2]. As in [2], let $\mathbf{X}_n = (V_n, E_n)$ be the colored graph which has the following vertices:

- $a_1, \ldots, a_n$,
- $b_1, \ldots, b_n$,
- m_S, for each subset S of $\{1, \ldots, n\}$ of even cardinality.
- The vertices a_i and b_i have the same color i and are called *mates* of each other,
- The vertices m_S are colored magenta (which is different from $1, \ldots, n$),

- If $i \neq j$, then a_i and a_j have different color,

and the following edges:

- a_i and m_S are joined by an edge if $i \in S$,
- b_i and m_S are joined by an edge if $i \notin S$.

The relevant properties of these graphs are ([2]):

- For any even number of pairs $\{a_i, b_i\}$ there is an automorphism of $\mathbf{X}_n$ which swaps a_i and b_i for these i and leaves other a_j and b_j fixed.
- Every automorphism of $\mathbf{X}_n$ is obtained in this way.

Let $n = l + 2$. Let $\mathbf{T}$ be a copy of $\mathbf{K}_n$, i.e. the complete graph with n vertices. We define a new graph $\mathbf{G}$ as follows: For each vertex v of $\mathbf{T}$, we replace v by a copy $\mathbf{X}(v)$ of $\mathbf{X}_{n-1}$. These copies are called *gadgets.* We endow $\mathbf{X}(v)$ with an additional color which is different for different v. For each $w \neq v$ in $\mathbf{T}$ we select one pair $\{a_{i(v,w)}, b_{i(v,w)}\}$ from $\mathbf{X}(v)$ and one pair $\{a_{i(w,v)}, b_{i(w,v)}\}$ from $\mathbf{X}(w)$, in such a way that all selected pairs in each gadget are distinct. Then we connect with an edge $a_{i(v,w)}$ to $a_{i(w,v)}$ and $b_{i(v,w)}$ to $b_{i(w,v)}$. This ends the description of $\mathbf{G}$. The graph $\mathbf{H}$ is defined similarly, except that for one edge (v_0, w_0) of $\mathbf{T}$, we create a "twist" by connecting $a_{i(v_0,w_0)}$ to $b_{i(w_0,v_0)}$ and $b_{i(v_0,w_0)}$ to $a_{i(w_0,v_0)}$.

Note that because of the coloring, an automorphism of $\mathbf{G}$ or $\mathbf{H}$ induces an automorphism of each gadget, and *vice versa.* The crucial properties of $\mathbf{G}$ and $\mathbf{H}$ are:

- For any even number of edges $\{(v_1, w_1), \ldots, (v_l, w_l)\}$ of $\mathbf{T}$ there is an automorphism of $\mathbf{G}$ ($\mathbf{H}$) which twists $\{(v_1, w_1), \ldots, (v_l, w_l)\}$ and leaves other edges fixed. Every automorphism of $\mathbf{G}$ and $\mathbf{H}$ is obtained in this way.
- For any edge (v, w) of $\mathbf{T}$ there is an isomorphism $\mathbf{G}^* \to \mathbf{H}$, where $\mathbf{G}^*$ is obtained from $\mathbf{G}$ by twisting (v, w) and leaving other edges fixed.
- $\mathbf{G}$ and $\mathbf{H}$ are not isomorphic.

Since f is unbounded, there is a natural number m so that $d \leq f(m) \leq m - d$, where $d = n(2^{n-2} + 2(n-1))$. Let $\mathbf{G}'$ be the disjoint union of $\mathbf{G}$ and two complete graphs $\mathbf{K}_u$ and $\mathbf{K}_v$, where

$$u = f(m) - n, \;\; v = m - f(m) - d + n.$$

Likewise, let $\mathbf{H}'$ be the disjoint union of $\mathbf{H}$, $\mathbf{K}_u$ and $\mathbf{K}_v$. In both graphs the clique $\mathbf{K}_u$ is colored beige and $\mathbf{K}_v$ violet. Note that the number of vertices of both $\mathbf{G}'$ and $\mathbf{H}'$ is exactly m.

The automorphisms of $\mathbf{G}'$ consist of an automorphism of $\mathbf{G}$ plus permutations of $\mathbf{K}_u$ and $\mathbf{K}_v$. The same holds for $\mathbf{H}'$. Thus all the automorphisms of $\mathbf{G}'$ and $\mathbf{H}'$ are known. The same is true of isomorphisms $\mathbf{G}'^* \to \mathbf{H}'$, where $\mathbf{G}'^*$ is obtained from $\mathbf{G}'$ by twisting an edge (v, w) of $\mathbf{T}$ and leaving other edges fixed.

We shall now show that $\mathbf{G}' \equiv_{\mathcal{L}^l_{\infty\omega}(\mathbf{Q}_1)} \mathbf{H}'$ by describing a winning strategy of Duplicator in $\mathrm{BP}^l_1(\mathbf{G}', \mathbf{H}')$. The task of Duplicator is to choose bijections $G' \to H'$. Suppose Duplicator has played f_{q-1}, Spoiler has played C_{q-1} and D_{q-1} and the referee has defined $p_{q-1} = (p_{q-2} \restriction C_{q-1}) \cup (f_{q-1} \restriction D_{q-1})$ which is indeed a partial isomorphism $\mathbf{G}' \to \mathbf{H}'$. (Let $p_0 = f_0 \restriction D_0$.) We assume that Duplicator found f_{q-1} by choosing a vertex v_{q-1} from $\mathbf{T}$, twisting an edge adjacent to it and letting f_{q-1} be the resulting isomorphim onto $\mathbf{H}'$. Naturally, v_{q-1} had to be chosen carefully. Now, Duplicator chooses $v_q \neq v_{q-1}$ so that $\mathbf{X}(v_q)$ does not meet $C_{q-1} \cup D_{q-1}$. This is possible because $n > l + 1$. Then he twists the edge (v_q, v_{q-1}) and lets f_q be the resulting isomorphism onto $\mathbf{H}'$ such that $p_{q-1} \subseteq f_q$. Next Spoiler plays C_q and D_q and the referee defines $p_q = (p_{q-1} \restriction C_q) \cup (f_q \restriction D_q)$. This is a partial isomorphism $\mathbf{G}' \to \mathbf{H}'$, because D_q is a singleton and no edge with one end in $dom(p_{q-1})$ was twisted when f_q was defined. This ends the description of the winning strategy of Duplicator in $\mathrm{BP}^k_1(\mathbf{G}', \mathbf{H}')$, thereby ending the proof of $\mathbf{G}' \equiv_{\mathcal{L}^l_{\infty\omega}(\mathbf{Q}_1)} \mathbf{H}'$.

We shall next describe the sentence of $\mathcal{L}_{\omega\omega}(\mathrm{Ram}^2(Q))$ that separates these two graphs. The sentence stipulates the existence of a set which contains all beige elements and exactly one element from the magenta part of each gadget $\mathbf{X}_{n-1}$. Once we have this one fixed element x in the magenta part of a gadget, we can divide each pair (a_i, b_i) into the element with an edge to x – call it "lower" – and the element without an edge to x – call it "upper" . The sentence we need says: There is a set X such that $|X| \geq f(m)$ and any two distinct elements x and y from X satisfy:

1. x is magenta or beige,

2. If x and y are not beige, then they are in different gadgets,

3. If there is a connection from the gadget of x to the gadget of y, it connects "upper" elements to "upper" elements.

To see that this sentence really separates $\mathbf{G}'$ and $\mathbf{H}'$ we first note that the sentence is true in $\mathbf{G}'$, since we can let X consist of all beige elements plus the element $m_\emptyset$ from all gadgets. Then $|X| = u + n = f(m)$. On the other hand, suppose such a set X existed in $\mathbf{H}'$. Since elements of X are either beige or in different gadgets, $|X| \leq u + n = f(m)$, whence actually $|X| = f(m)$. Thus X consists of exactly the beige elements and exactly one vertex from the magenta part of each gadget. This element has the form m_S for some set S of even cardinality, so the a_i for $i \in S$ are "lower". Now, in each gadget, flip the even number of pairs $(a_i, b_i), i \in S$, but leave the others. This results in an automorphism h of $\mathbf{H}'$ such that, in each gadget, $h(m_S) = m_\emptyset$. But with respect to $m_\emptyset$, all the a_i are "upper". Since part 3 of the sentence is true (for $h(x)$ and $h(y)$), it follows that $\mathbf{H}'$ must be isomorphic to $\mathbf{G}'$, which is a contradiction. Thus the sentence is false in $\mathbf{H}'$, and the proof is finished. □

4.4 Theorem. *If f is unbounded, then* $\mathrm{Ram}^{k+1}(Q_f)$ *is not definable in* $\mathcal{L}^\omega_{\infty\omega}(\mathbf{Q}_k)$.

Proof. The case $k = 1$ is Theorem 4.3, so we assume $k > 1$. Fix $l \geq 1$. We use the models $\mathbf{A} = \mathbf{A}(\mathbf{G})$ and $\mathbf{B} = \mathbf{B}(\mathbf{G})$ constructed in [7]. By general results in [7] we have $\mathbf{A} \equiv_{\mathcal{L}^l_{\infty\omega}(\mathbf{Q}_k)} \mathbf{B}$. By [7, Corollary 8.8] $\mathbf{A}$ and $\mathbf{B}$ can be separated by $\mathrm{Ram}^{k+1}(Q_f)$ for $f(n) = [n/2]$. We shall make a little modification to the proof of [7, Corollary 8.8] in order to cover an arbitrary unbounded f.

Let x be the size of the graph $\mathbf{G}$. Note that $\mathbf{A}$ has a homogeneous subset of cardinality $x(k+1)$ while $\mathbf{B}$ does not. With this in mind, and since f is unbounded, we choose an n with $x(k+1) \leq f(n) \leq n - x(k+1)$. Note that $|\mathbf{A}| = 2x(k+1)$. We extend the model $\mathbf{A}$ with two disjoint sets U and V, where $|U| = f(n) - x(k+1)$ and $|V| = n - x(k+1) - f(n)$. The elements of U are colored beige and the elements of V violet. The same extension is applied to $\mathbf{B}$. We get two new models $\mathbf{A}'$ and $\mathbf{B}'$ of size

n. Clearly still $\mathbf{A}' \equiv_{\mathcal{L}^l_{\infty\omega}(\mathbf{Q}_k)} \mathbf{B}'$. We can separate these models with the sentence $\mathrm{Ram}^{k+1}(Q)x_1, \ldots, x_{k+1}\psi'(x_1, \ldots, x_{k+1})$, where $\psi'(x_1, \ldots, x_{k+1})$ is the disjunction of $\psi(x_1, \ldots, x_{k+1})$ from [7, Corollary 8.8] and the formula "x_1 is beige". □

Combining Theorems 2.3 and 4.4 we obtain a complete characterization of the definability of Ramseyfication for all monotone simple monadic quantifiers. And as in the case of branching, the same characterization holds also for relativizations of such quantifiers (this follows from Theorem 2.5 and the fact that $\mathrm{Ram}^k(Q_f)$ is definable in $\mathcal{L}_{\omega\omega}(\mathrm{Ram}^k(Q_f^{\mathrm{rel}}))$).

4.5 Corollary. *Suppose that Q is either a monotone type $\langle 1\rangle$ quantifier, or a right monotone type $\langle 1, 1\rangle$ quantifier satisfying CONS and EXT. Then the following conditions are equivalent, for any $k \geq 1$:*
(1) $\mathrm{Ram}^{k+1}(Q)$ *is definable in* $\mathcal{L}_{\omega\omega}(Q)$.
(2) $\mathrm{Ram}^{k+1}(Q)$ *is definable in* $\mathcal{L}^{\omega}_{\infty\omega}(\mathbf{Q}_k)$.
(3) Q *is bounded.*

5 Definability of resumption

We do not have as strong results for $\mathrm{Res}^k(Q)$ as for $\mathrm{Br}^k(Q)$ and $\mathrm{Ram}^{k+1}(Q)$ above. In fact, it is proved in [11] that for type $\langle 1\rangle$ Q, $\mathrm{Res}^k(Q)$ is always definable in $\mathcal{L}^{\omega}_{\infty\omega}(\mathbf{Q}_1)$ (this extends to resumptions of any monadic quantifiers). But we prove that if f is unbounded in a sense which is appropriate from the resumption point of view, then $\mathrm{Res}^k(Q_f)$ is not definable in $\mathcal{L}^{\omega}_{\infty\omega}(Q_1, \ldots, Q_m)$ for any $Q_1, \ldots, Q_m \in \mathbf{M}$ and hence not in $\mathcal{L}_{\omega\omega}(\mathbf{M})$, where $\mathbf{M}$ is the collection of all monotone type $\langle 1\rangle$ quantifiers. For this result we need again a new version of an Ehrenfeucht-Fraïssé type game from [11].

Let $\mathbf{Q} = \{Q_i \mid i \in I\}$ be a family of type $\langle 1\rangle$ quantifiers, $\mathbf{A}$ and $\mathbf{B}$ two structures, and l a positive integer. The game $\mathrm{MEF}^l(\mathbf{Q}, \mathbf{A}, \mathbf{B})$ has the following rules: There are l pairs of pebbles. During the game Spoiler may take a pebble and put it on an element of one of the models, and then Duplicator has to take the corresponding pebble and put it on an element of the other model. If all pebbles have been used already, Spoiler can remove one of his pebbles from one of the models and reposition it on another element of one of the model. In such a case Duplicator has to reposition the corresponding pebble on some element of the other model. These pebble moves are called elementary moves.

There is another type of move that Spoiler can choose to make. This is called a quantifier move. In this move Spoiler takes a pebble, one of the structures $\mathbf{A}$ and $\mathbf{B}$, say $\mathbf{A}$, a quantifier Q_j from the family $\mathbf{Q}$, and a subset X of the universe A of $\mathbf{A}$ such that the structure (A, X) is in the quantifier Q_j, and the set X is invariant under all automorphisms of $\mathbf{A}$ that fix all pebbled elements. Duplicator must respond by choosing a subset Y of the universe of the other structure (in this case Y must be a subset of the universe B of $\mathbf{B}$) such that the structure (B, Y) is in the quantifier Q_j. Then Spoiler places the pebble on an element $b_1 \in Y$ and Duplicator must respond by placing a pebble on an element $a_1 \in X$.

In each round of the game Spoiler first makes an elementary move or a quantifier move and then Duplicator responds as described above. This is repeated indefinitely. In each round, if a_i and b_i, $1 \leq i \leq l', l' \leq l$, are the elements of $\mathbf{A}$ and $\mathbf{B}$ respectively pebbled by the two players using the up to l pebbles available, then Spoiler wins if the mapping $a_i \mapsto b_i$, $1 \leq i \leq l'$, fails to be a partial isomorphism between $\mathbf{A}$ and $\mathbf{B}$. Otherwise, the game goes on. If the game lasts for infinitely many moves without Spoiler winning, then Duplicator is declared the winner.

5.1 Proposition. ([11]) *Let $\mathbf{Q} = \{Q_i \mid i \in I\}$ be a family of monotone type $\langle 1 \rangle$ quantifiers, $\mathbf{A}$ and $\mathbf{B}$ two finite structures, and l a positive integer. Then the following statements are equivalent (where $\mathcal{L}^l_{\omega\omega}$ is like $\mathcal{L}_{\omega\omega}$ but with only l variables):*

(1) $\mathbf{A} \equiv_{\mathcal{L}^l_{\omega\omega}(\mathbf{Q})} \mathbf{B}$

(2) $\mathbf{A} \equiv_{\mathcal{L}^l_{\infty\omega}(\mathbf{Q})} \mathbf{B}$

(3) *Duplicator has a winning strategy for the game* $\mathrm{MEF}^l(\mathbf{Q}, \mathbf{A}, \mathbf{B})$.

5.2 Corollary. ([11]) *Let Q be a quantifier and $\mathbf{Q}$ a family of monotone type $\langle 1 \rangle$ quantifiers. If for some sentence φ of $\mathcal{L}_{\omega\omega}(Q)$ and for all natural numbers l there exist structures $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{A} \models \varphi$, $\mathbf{B} \not\models \varphi$ and Duplicator has a winning strategy in $\mathrm{MEF}^l(\mathbf{Q}, \mathbf{A}, \mathbf{B})$, then Q is not definable in $\mathcal{L}^\omega_{\infty\omega}(\mathbf{Q})$.*

We shall now give some examples concerning the definability of $\mathrm{Res}^k(Q_f)$ in $\mathcal{L}_{\omega\omega}(Q_f)$. The examples display characteristic features of resumption in this respect.

5.3 Proposition. *Both boundedness of f and unboundedness of f are consistent with* $\mathrm{Res}^k(Q_f)$ *being definable in* $\mathcal{L}_{\omega\omega}(Q_f)$, *but neither condition is sufficient or necessary for this to hold.*

Proof. Suppose $f(n) = 1$ if n is a k'th power of an integer and $f(n) = 2$ otherwise. Then f is bounded and Q_f is not first order (cf. Theorem 2.1), but $\mathrm{Res}^k(Q_f)$ is. Hence $\mathrm{Res}^k(Q_f)$ is trivially definable from Q_f. Thus boundedness is consistent with $\mathrm{Res}^k(Q_f)$ being definable in $\mathcal{L}_{\omega\omega}$ even when Q_f is not.

To see that boundedness is not sufficient for $\mathrm{Res}^k(Q_f)$ to be definable from Q_f, suppose $f(n) = 1$ if n is the k'th power of a prime and $f(n) = 2$ otherwise. Then f is bounded and Q_f is not first order. Moreover, $\mathrm{Res}^k(Q_f)$ is not definable from Q_f, as the following argument shows: We can use $\mathrm{Res}^k(Q_f)$ to define the property "the size of the universe is a prime". But we can use Corollary 5.2 to show that this property is not definable from Q_f: Suppose l is a natural number. Let $p > l$ be a prime such that $p + 1 \neq 2^k$. Then Duplicator wins the game $\mathrm{MEF}^l(Q_f, \{1, \ldots, p\}, \{1, \ldots, p + 1\})$.

Unboundedness of f is consistent with $\mathrm{Res}^k(Q_f)$ being definable in $\mathcal{L}_{\omega\omega}$ even when Q_f is not. To see this, suppose $f(n) = 1$ if n is a k'th power and $f(n) = [n/2]$ otherwise. Then f is unbounded and Q_f is not first order, but $\mathrm{Res}^k(Q_f)$ is.

Finally, suppose $f(n) = 1$ if n is the k'th power of a prime and $f(n) = [n/2]$ otherwise. Then f is unbounded, Q_f is not first order, and $\mathrm{Res}^k(Q_f)$ is not definable from Q_f, because "the size of the universe is a prime" is definable from $\mathrm{Res}^k(Q_f)$, but not from Q_f. □

The next result shows that there are reasonable conditions for f which guarantee that $\mathrm{Res}^k(Q_f)$ is definable in terms of Q_f.

Let f and g be functions $\omega \to \omega$. We say that g is *determined* by f, if for all $m, n \in \omega$, $f(n) = f(m) \implies g(n) = g(m)$. Furthermore, f is *upwards bounded* if there exists t such that $f(n) < t$ for all n.

5.4 Proposition. *Suppose f is upwards bounded and the function $g : n \mapsto f(n^k)$ is determined by f. Then* $\mathrm{Res}^k(Q_f)$ *is definable in* $\mathcal{L}_{\omega\omega}(Q_f)$.

Proof. Let ξ and φ_p $(p > 0)$ be as in the proof of Theorem 2.3, and let $\varphi_0 = \xi$. Thus, φ_p says that $f(|A|) = p$. For $i \geq 0$ let Φ_i be a sentence in $\mathcal{L}_{\omega\omega}$ saying of a k-ary predicate R that the number of k-tuples it contains is at least i. By assumption there is a function h such that $f(n^k) = h(f(n))$. Let t be an upper bound for f, i.e., $f(n) < t$ for all n. Now $\mathrm{Res}^k(Q_f)x_1, \ldots, x_k R(x_1, \ldots, x_k)$ is equivalent to the $\mathcal{L}_{\omega\omega}(Q_f)$-sentence $\bigvee_{i<t}(\varphi_i \wedge \Phi_{h(i)})$. □

The condition of Proposition 5.4 holds, for example, for the following f:

$$f_p(n) = \begin{cases} 0 & \text{if } p \text{ divides } n \\ 1 & \text{otherwise.} \end{cases}$$

Thus $\mathrm{Res}^k(Q_{f_p})$ is definable from Q_{f_p}. By a result of M. Mostowski ([17]), resumptions of the stronger divisibility quantifiers

$$D_p x\phi(x) \iff p \text{ divides the number of elements } x \text{ satisfying } \phi(x).$$

are likewise definable from the quantifiers themselves.

5.5 Definition. Let $X \subseteq \omega$. A function $f : \omega \to \omega$ is *unbounded on* X, if $\forall m \exists n \in X(m \leq f(n) \leq n - m)$. Otherwise f is *bounded on* X.

The following theorem is the main result of this section:

5.6 Theorem. *Suppose f is unbounded on $\{n^k \mid n < \omega\}$, where $k \geq 2$. Then $\mathrm{Res}^k(Q_f)$ is not definable in $\mathcal{L}^{\omega}_{\infty\omega}(Q_1, \ldots, Q_m)$ for any $Q_1, \ldots, Q_m \in$ $\mathbf{M}$.*

Proof. For every m there is an n so that $m \leq f(n^k) \leq n^k - m$. This $f(n^k)$ is either $\leq [n^k/2]$ or $> [n^k/2]$. One of these possibilities occurs infinitely often, and hence we have

$$\forall m \exists n((m \leq f(n^k) \leq \left[\frac{n^k}{2}\right]) \tag{1}$$

or

$$\forall m \exists n(\left[\frac{n^k}{2}\right] < f(n^k) \leq n^k - m)). \tag{2}$$

We start by assuming (1) and indicate at the end of the proof how to handle the case that (2) holds.

Suppose $g_1, \ldots, g_m$ are functions and l is a natural number. We are going to prove that $\mathrm{Res}^k(Q_f)$ is not definable in $\mathcal{L}^{l}_{\infty\omega}(Q_{g_1}, \ldots, Q_{g_m})$. For any $n \geq l$ let $\mathcal{I}_n$ consist of intervals of the following form:

- $[0, l]$,
- $[n-l, n]$,
- $[g_i(n)-l-1, g_i(n)+l+1]$, where $i = 1, \ldots, m$,
- $[n-g_i(n)-l-1, n-g_i(n)+l+1]$, where $i = 1, \ldots, m$.

We call a real $a \in [0, n]$ *n-good*, if a is not on any of the intervals in $\mathcal{I}_n$. A set $X \subseteq [0, n]$ is n-good if each of its elements is. A direct calculation reveals that if I is a subinterval of $[0, n]$ of length $\geq C(x) = (2m+1)x + 2(2m+1)(l+1)$, then there is an n-good subinterval J of I of length x. Let $C_1 = C(1)$ and $C_2 = C(C_1)$.

Let $E = (C_2 + C_2/(\sqrt[k(k-1)]{2} - 1))^k$. By (1) there is an n such that $E \leq \lambda = f(n^k) \leq n^k/2$. Let $\mu = \sqrt[k]{\lambda}$ and $\nu = \sqrt[k-1]{\lambda/n}$.

Claim 1. $\mu - \nu \geq C_2$.

Let $D = (C_2/(\sqrt[k(k-1)]{2} - 1))^{k-1}$. If $\lambda \geq Dn$, then

$$\mu - \nu = \nu(\sqrt[k(k-1)]{\frac{n^k}{\lambda}} - 1) \geq \sqrt[k-1]{\frac{\lambda}{n}}(\sqrt[k(k-1)]{2} - 1) \geq \sqrt[k-1]{D}(\sqrt[k(k-1)]{2} - 1) = C_2.$$

If $\lambda < Dn$, then

$$\mu - \nu \geq \sqrt[k]{E} - \sqrt[k-1]{D} = C_2 + \frac{C_2}{(\sqrt[k(k-1)]{2} - 1)} - \sqrt[k-1]{D} = C_2.$$

In either case we have proved the claim.

Claim 2. There are an n-good x and a non-integer n-good $y > x$ on $[0, n]$ so that $x^{k-1} \cdot y = \lambda$.

By Claim 1, there is an n-good subinterval I of $[\nu, \mu]$ of length $> C_1$. On the interval $[\nu, \mu]$ the function $F(x) = \lambda x^{1-k}$ is one-one onto $[\mu, n]$ with derivative $F'(x) < -1$, and so the interval $F''I$ has length $> C_1$. Hence there is a non-integer n-good $y = F(x)$ on $F''I$. Now also x is n-good and $x^{k-1} \cdot y = \lambda$. Claim 2 is proved.

Let a be the integer part of x, and b the integer part of y. We define now two structures $\mathbf{A} = (A, P, R)$ and $\mathbf{B} = (A, P', R')$ as follows:

- $A = \{1, \ldots, n\}$
- $P = \{1, \ldots, a+1\}$, $P' = \{1, \ldots, a\}$

- $R = \{1, \ldots, b+1\}$, $R' = \{1, \ldots, b\}$.

Claim 3. Duplicator wins the game $\mathrm{MEF}^l(\{Q_{g_1}, \ldots, Q_{g_m}\}, \mathbf{A}, \mathbf{B})$.

The elementary moves are easy for Duplicator. Suppose then Spoiler chooses a pebble, the model $\mathbf{A}$, the quantifier Q_{g_j}, and an invariant subset X of A with $|X| \geq g_j(n)$. Suppose furthermore that the pebbled elements in A are $a_1, \ldots, a_s$, where $s < l$, and that Duplicator has not lost the game yet. Let the correspondingly pebbled elements of $\mathbf{B}$ be $b_1, \ldots, b_s$. For any set Z and $d \in \{0,1\}$ let $Z^0 = \emptyset$ and $Z^1 = Z$. Since X is invariant under automorphisms of $\mathbf{A}$ that fix the elements $a_1, \ldots, a_s$, there are $d_1, d_2, d_3 \in \{0,1\}$ so that

$$(X - X^\star) \cup (X^\star - X) \subseteq \{a_1, \ldots, a_s\},$$

where

$$X^\star = P^{d_1} \cup (R-P)^{d_2} \cup (A-R)^{d_3}.$$

Let

$$Y^\star = P'^{d_1} \cup (R'-P')^{d_2} \cup (A-R')^{d_3}$$

and

$$Y = (Y^\star \cup \{b_r \mid a_r \in X\}) - \{b_r \mid a_r \notin X\}.$$

If $t = |X| - |X^\star|$, then $t = |Y| - |Y^\star|$. Moreover, $|X^\star| = |Y^\star|$ and $|X| = |Y|$, unless $|Y^\star| \in \{a, b, n-a, n-b\}$ but even then $||X^\star| - |Y^\star|| \leq 1$ and $||X| - |Y|| \leq 1$. Therefore we can make the following inference, making use of the n-goodness of x and y: If $|Y| < g_j(n)$, then $|Y^\star| < g_j(n) - t$. Hence $|X^\star| < g_j(n) - t$ and $|X| < g_j(n)$, contrary to assumption. This inference shows that $|Y| \geq g_j(n)$. So Duplicator can play the set Y. Next Spoiler puts the pebble on some element y of Y. It is now easy for Duplicator to put his pebble on an element x of X so that the following condition are satisfied:

- $x = a_i$ if and only if $y = b_i$ for $i = 1, \ldots, s$.
- $x \in P$ if and only if $y \in P'$.
- $x \in R$ if and only if $y \in R'$.

Claim 3 is proved.

To end the proof we have to exhibit a sentence of $\mathcal{L}_{\omega\omega}(\mathrm{Res}^k(Q_f))$ which distinguishes $\mathbf{A}$ and $\mathbf{B}$. Consider the sentence $\varphi = \mathrm{Res}^k(Q_f)x_1, \ldots, x_k(P(x_1) \wedge$

$\ldots \wedge P(x_{k-1}) \wedge R(x_k))$ of $\mathcal{L}_{\omega\omega}(\mathrm{Res}^k(Q_f))$. This sentence is true in $\mathbf{A}$ but false in $\mathbf{B}$ because

$$a^{k-1} \cdot b < f(n^k) = x^{k-1} \cdot y \leq (a+1)^{k-1} \cdot (b+1)$$

We made the assumption that (1) holds. Suppose now (2) holds. We proceed as above until the definition of n and λ. This time we use (2) to find an n so that $[n^k/2] < f(n^k) \leq n - E$ and let $\lambda = n^k - f(n^k) + 1$. Now $E \leq \lambda \leq n^k/2$, so we can continue as above. When we come to the sentence φ, we replace it with

$$\neg\mathrm{Res}^k(Q_f)x_1, \ldots, x_k(\neg(P(x_1) \wedge \ldots \wedge P(x_{k-1}) \wedge R(x_k)).$$

The theorem is proved. □

A special case of the above theorem — that $\mathrm{Res}^2(Q_f)$ is not definable in $\mathcal{L}_{\omega\omega}(Q_f)$ for $f(n) = [n/2]$ — was proved in [20].

5.7 Theorem. *The following conditions are equivalent for any monotone simple monadic quantifier Q and any $k \geq 2$:*
(1) $\mathrm{Res}^k(Q)$ *is definable in* $\mathcal{L}_{\omega\omega}(\mathbf{M})$.
(2) Q *is bounded on* $\{n^k \mid n < \omega\}$.

Proof. Theorem 5.6 gives (1) $\Rightarrow$ (2). Assume then (2). So there is a number m such that for all n, either

$$g(n) = f(n^k) < m$$

or

$$h(n) = n^k - f(n^k) < m.$$

Let θ_i (χ_i) be a sentence in $\mathcal{L}_{\omega\omega}(Q_g)$ ($\mathcal{L}_{\omega\omega}(Q_h)$) saying that $g(|A|) = i$ ($h(|A|) = i$), and let Φ_i and Ψ_i be as in the proof of Proposition 5.4. Also, let φ, ψ be $\mathcal{L}_{\omega\omega}$-sentences such that φ says that the universe has size $< m$ and ψ defines $\mathrm{Res}^k(Q_f)$ on such universes. Then the sentence $\mathrm{Res}^k(Q_f)x_1, \ldots, x_k R(x_1, \ldots, x_k)$ is equivalent to the $\mathcal{L}_{\omega\omega}(Q_g, Q_h)$-sentence

$$(\varphi \wedge \psi) \ \vee \ (\neg\varphi \ \wedge \ (\bigvee_{i<m} (\theta_i \wedge \Phi_i) \ \vee \ \bigvee_{i<m} (\chi_i \wedge \Psi_i))).$$

□

6 Conclusion

The field of definability of generalized quantifiers on finite structures is full of open problems. Especially the polyadic quantifiers as well as the non-monotone ones give rise to problems that no one knows how to approach. In this paper we have considered certain polyadic lifts of monotone quantifiers, obtaining the following results ($k \geq 2$):

I. $\mathrm{Br}(Q_{f_1}, \ldots, Q_{f_k})$ is definable in $\mathcal{L}_{\omega\omega}(Q_{f_1}, \ldots, Q_{f_k})$
iff $\mathrm{Br}(Q_{f_1}, \ldots, Q_{f_k})$ is definable in $\mathcal{L}_{\omega\omega}(\mathbf{Q}_1)$
iff $\langle f_1, \ldots, f_k \rangle$ is bounded.

II. $\mathrm{Ram}^k(Q_f)$ is definable in $\mathcal{L}_{\omega\omega}(Q_f)$
iff $\mathrm{Ram}^k(Q_f)$ is definable in $\mathcal{L}^{\omega}_{\infty\omega}(\mathbf{Q}_{k-1})$
iff f is bounded.

III. $\mathrm{Res}^k(Q_f)$ is definable in $\mathcal{L}_{\omega\omega}(\mathbf{M})$ iff f is bounded on $\{n^k \mid n < \omega\}$.

I and II were also proved for relativizations of monotone simple unary quantifiers. The following questions seem to be the most natural direction to continue the work of this paper:

1. Find necessary and sufficient conditions for $\mathrm{Br}(Q_{f_1}, \ldots, Q_{f_k})$ to be definable in $\mathcal{L}^{\omega}_{\infty\omega}(\mathbf{Q}_1)$.
2. Find necessary and sufficient conditions for $\mathrm{Res}^k(Q)$ to be definable in $\mathcal{L}_{\omega\omega}(Q)$. This question makes sense for non-monadic and non-monotone quantifiers, too.
3. Find necessary and sufficient conditions for $\mathrm{Res}^2(Q_f)$ to be definable in $\mathcal{L}_{\omega\omega}(\mathbf{Q}_1)$. (As we mentioned in the Introduction, K. Luosto [15] has proved that $\mathrm{Res}^2(Q_f)$ is not definable in $\mathcal{L}_{\omega\omega}(\mathbf{Q}_1)$ for $f(n) = [n/2]$.)
4. Find a suitable extension of III to relativizations of monotone simple unary quantifiers.

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